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APPLIED MATHEMATICS RESEARCH GROUP RALEIGH

Fractional Integration And
Dual Integral Equations

by

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Contract No. Nonr 486(06)
File No. PSR-6

June 25, 1962

SCHOOL OF PHYSICAL SCIENCES AND APPLIED MATHEMATICS

(5) 643 550

NORTH CAROLINA STATE COLLEGE
DEPARTMENTS OF MATHEMATICS AND ENGINEERING RESEARCH
RALEIGH

(6) Lectures on Fractional Integration and
Dual Integral Equations

(9) June 25, 1962

(7) NA

(10) 40 p. 21 refs.

(11) Rept. no. PSR-6

(12) acc. cover

(13) NA

(14) Uncl

(15) NA

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PREFACE

This project report is based on six lectures given by Dr. I. N. Sneddon at North Carolina State College in the spring of 1962. The results of the research reported here have been successfully applied to solution of certain crack problems in the mathematical theory of elasticity; these applications are expected to appear either as a subsequent project report or in published journals.

Copies of this report have been distributed as directed by the sponsoring organizations. This project is sponsored by AFOSR, ARO, and ONR through the Joint Services Advisory Group; the contract is currently under ONR (Nonr 486(06)).

John W. Cell
Project Director

CONTENTS

LECTURES ON FRACTIONAL INTEGRATION AND DUAL INTEGRAL EQUATIONS

1. Introduction.

These notes are based on six lectures given in the Department of Mathematics, North Carolina State College, Raleigh, the final copy being prepared with the assistance of Dr. M. Lowengrub. Their function is purely expository; they contain little that is new but it was thought to be worthwhile to cast the material of Erdelyi and Sneddon (1962) into a form in which it could be applied immediately to some of the dual integral equations arising in mathematical physics. This is achieved by a trivial alteration in the definitions of the Erdelyi-Kober operators (cf. equations (3.1) and (3.2) below) and the modified operator of Hankel transforms.

The plan of the lectures is straightforward. First of all an account is given of the "classical" fractional integrals of Riemann-Liouville and Weyl (§2) and then of the closely related Erdelyi-Kober operators. One of the disadvantages of the form of definition of these operators adopted here is that whereas in the papers of Erdelyi and Kober certain formulae involved the simple differential operator $D = d/dx$, the analogous formulae here involve the more cumberson operator $\mathcal{D}_x = \frac{1}{2} Dx^{-1}$. The central part of the method is contained in § 4 where we derive relations connecting the Erdelyi-Kober operators with the modified operator of Hankel transforms. These relations are used extensively in applications so we have collected them together in Table I at the end. In the next section (§ 5) we introduce two operators $P^k_{\eta,\alpha}$, $Q^k_{\eta,\alpha}$ (defined by equations (5.1) and (5.2) respectively) and derive two of their properties; since these properties depend on results in the theory of Bessel functions due to Sonine, we have called them Sonine operators, although their use was suggested by a recent paper of Peters.

The remaining sections are concerned with applications. In § 6 we consider the type of pair of dual integral equations which arise in the analysis of mixed boundary value problems of potential theory concerning a half-space. In § 7 we consider the type arising in some problems of diffraction theory, and, finally in § 8, the more complicated type such as those arising in the solution of mixed boundary values of potential theory relating to a thick plate.

In deriving the basic results it is necessary first to consider some simple integrals. In order not to complicate the later proofs we discuss these integrals now. The first one is

$$2\int_v^x (x^2-u^2)^{\alpha-1}(u^2-v^2)^{\beta-1}\, u^{1-2\alpha-2\beta}du = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}\, v^{-2\alpha}x^{-2\beta}(x^2-v^2)^{\alpha+\beta-1} \tag{1.1}$$

in which

$$0<v<x, \quad \operatorname{Re}\alpha>0, \quad \operatorname{Re}\beta>0.$$

The proof is simple: If we change the variable of integration from <u>u</u> to <u>z</u> where

$$\frac{1}{u^2} = \frac{1}{v^2}\left\{1-\frac{x^2-v^2}{x^2}\,z\right\}$$

we find that the integral is transformed to

$$\int_0^1 \left(\frac{x^2-v^2}{v^2}\right)^{\alpha-1}(1-z)^{\alpha-1}\left(\frac{x^2-v^2}{x^2}\right)^{\beta-1} z^{\beta-1}\left(\frac{x^2-v^2}{x^2\;v^2}\right)dz$$

from which the result follows immediately.

Similarly, if $v>x>0$, $\operatorname{Re}\alpha>0$, $\operatorname{Re}\beta>0$, we have that

$$2\int_x^v (u^2-x^2)^{\alpha-1}(v^2-u^2)^{\beta-1}\,u\,du = (v^2-x^2)^{\alpha+\beta-1}\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}, \tag{1.2}$$

a result which is easily established by changing the variable of integration from <u>u</u> to <u>z</u> where

$$u^2 = x^2+(v^2-x^2)z.$$

We also require some results from the theory of Bessel functions. As a special case of the Weber-Schaftheitlin integral (Watson, 1944, p. 398) we have that if $\alpha+\beta>0$

$$\int_0^\infty v^{1-\alpha-\beta}J_{2\eta+2\alpha+\beta}(v)J_{2\eta+\alpha}(\rho v)dv = \begin{cases}\dfrac{\rho^{2\eta+\alpha}(1-\rho^2)^{\alpha+\beta-1}}{2^{\alpha+\beta-1}\Gamma(\alpha+\beta)}, & 0<\rho<1;\\[1ex] 0, & \rho>1.\end{cases}$$

Making the substitutions $v=xy$, $\rho=u/x$ we see that this result can be written in the equivalent form

$$2^{\alpha+\beta}\int_0^\infty y^{1-\alpha-\beta}J_{2\eta+2\alpha+\beta}(xy)J_{2\eta+\alpha}(uy)dy = \frac{2}{\Gamma(\alpha+\beta)}\,x^{-2\eta-2\alpha-\beta}u^{2\eta+\alpha}(x^2-u^2)^{\alpha+\beta-1}H(x-u) \tag{1.3}$$

where H denotes Heaviside's unit function.

A well-known result in the theory of Bessel functions is Sonine's first integral

$$J_{\mu+\nu+1}(z) = \frac{z^{\nu+1}}{2^{\nu}\,\Gamma(\nu+1)}\int_0^{\frac{1}{2}\pi} J_\mu(z\sin\theta)\sin^{\mu+1}\theta\cos^{2\nu+1}\theta\,d\theta.$$

(Watson, 1944, p.373). If we make the substitutions $z = yx$, $\sin\theta = u/x$ we find that we can write this result in the form

$$\int_0^x u^{\mu+1}(x^2-u^2)^{\nu}\,J_\mu(uy)du = 2^{\nu}x^{\mu+\nu+1}\,y^{-\nu-1}\,\Gamma(\nu+1)J_{\mu+\nu+1}(xy) \qquad (1.4)$$

Sonine's second integral states that if $\operatorname{Re}\nu > \operatorname{Re}\mu > -1$,

$$\int_0^\infty (t^2+z^2)^{-\frac{1}{2}\nu}\,t^{\mu+1}\,J_\mu(xt)J_\nu\left\{a\sqrt{(t^2+z^2)}\right\}dt$$
$$= a^{-\nu}x^{\mu}z^{-\nu+\mu+1}(a^2-x^2)^{\frac{1}{2}\nu-\frac{1}{2}\mu-\frac{1}{2}}\,J_{\nu-\mu-1}\left\{z\sqrt{(a^2-x^2)}\right\}H(a-x), \qquad (1.5)$$

(Watson, 1944, p. 415). The substitutions $t = \sqrt{(\tau^2-z^2)}$, $\mu = \nu - r - 1$ transform this equation to

$$\int_z^\infty (\tau^2-z^2)^{\frac{1}{2}\nu-\frac{1}{2}r-\frac{1}{2}}\,\tau^{-\nu+1}J_{\nu-r-1}\left\{x\sqrt{(\tau^2-z^2)}\right\}J_\nu(a\tau)d\tau$$
$$= a^{-\nu}x^{\nu-r-1}z^{-r}(a^2-x^2)^{\frac{1}{2}r}\,J_r\left\{z\sqrt{(a^2-x^2)}\right\}H(a-x), \qquad (1.6)$$

provided $\operatorname{Re}\nu > \operatorname{Re}\,\underline{r} > -1$. If we now divide both sides of this equation by $x^{\nu-r-1}$ and let x tend to zero, we find that

$$\int_z^\infty (\tau^2-z^2)^{\nu-r-1}\,\tau^{-\nu+1}\,J_\nu(a\tau)d\tau = 2^{\nu-r-1}\Gamma(\nu-r)a^{r-\nu}z^{-r}\,J_r(az), \qquad (1.7)$$

provided that $\operatorname{Re}\,\underline{a} > 0$, $\operatorname{Re}\nu > \operatorname{Re}\,\underline{r} > \operatorname{Re}(\frac{1}{2}\nu - \frac{3}{4})$.

If we apply Hankel's inversion theorem to equation (1.5) we obtain the equation

$$\int_0^a x^{\mu+1}(a^2-x^2)^{\frac{1}{2}\nu-\frac{1}{2}\mu-\frac{1}{2}}\,J_{\nu-\mu-1}\left\{z\sqrt{(a^2-x^2)}\right\}J_\mu(xt)dx$$
$$= a^{\nu}z^{\nu-\mu-1}t^{\mu}(t^2+z^2)^{-\frac{1}{2}\nu}\,J_\nu\left\{a\sqrt{(t^2+z^2)}\right\},\ (\operatorname{Re}\nu > \operatorname{Re}\,\underline{\mu} > -1). \qquad (1.8)$$

Replacing s by ik we obtain the equation

$$\int_0^a x^{\mu+1}(a^2 - x^2)^{\frac{1}{2}\nu - \frac{1}{2}\mu - \frac{1}{2}} I_{\nu-\mu-1}\left\{k\sqrt{(a^2 - x^2)}\right\} J_\mu(xt)dx$$

$$= a^\nu k^{\nu-\mu-1} t^\mu (t^2 - k^2)^{-\frac{1}{2}\nu} J_\nu\left\{a\sqrt{(t^2 - k^2)}\right\}. \qquad (1.9)$$

with the same restrictions on ν and μ.

2. Fractional Integration.

2.1 Repeated indefinite integrals.

If the function $f(x)$ is integrable in any interval $(0, a)$, where $a > 0$, we define the first integral $F_1(x)$ of $f(x)$ by the formula

$$F_1(x) = \int_0^x f(t)dt, \qquad (2.1)$$

and the subsequent integrals by the recursion formula

$$F_{r+1}(x) = \int_0^x F_r(t)dt \qquad (2.2)$$

where r is a positive integer. These integrals are such that

$$F_r(0) = 0 \qquad (r = 1, 2, \ldots), \qquad (2.3)$$

and have the property

$$F'_{r+1}(x) = F_r(x), \qquad (r = 1, 2, \ldots). \qquad (2.4)$$

Now if we consider the integral

$$\int_0^x (x - t)f(t)dt$$

we find, on integrating by parts, that it is equal to

$$[F_1(t)(t - x)]_0^x + \int_0^x F_1(t)dt = F_2(x),$$

and similarly that

$$\frac{1}{2!}\int_0^x (x - t)^2 f(t)dt = \frac{1}{2}[F_1(t)(x - t)^2]_0^x + [F_2(t)(x - t)]_0^x + \int_0^x F_2(t)dt$$

$$= F_3(x).$$

This leads us to speculate that

$$F_{n+1}(x) = \frac{1}{n!}\int_0^x (x - t)^n f(t)dt, \tag{2.5}$$

a result which is easily proved by induction.

Similarly we could define an indefinite integral $F_1^*(x)$ of $f(x)$ by the formula

$$F_1^*(x) = -\int_x^\infty f(t)dt \tag{2.6}$$

and the higher order integrals of the same family by the recursion formula

$$F_{r+1}^*(x) = -\int_x^\infty F_r(t)dt \tag{2.7}$$

These integrals also have the properties (2.3) and (2.4). It is easily proved (again by induction) that

$$F_{n+1}^*(x) = \frac{1}{n!}\int_x^\infty (t - x)^n f(t)dt \tag{2.8}$$

provided that $f(x)$ is of such a nature that the integral exists.

2.2 The Riemann-Liouville fractional integral.

The Riemann-Liouville fractional integral is a generalisation of the integral on the right-hand side of equation (2.5). The integral

$$\frac{1}{\Gamma(\alpha)}\int_0^x f(t)(x - t)^{\alpha-1}dt \tag{2.9}$$

is convergent for a wide class of functions $f(t)$ if Re $\alpha > 0$. The upper limit of integration x may be real or complex; in the latter case the path of integration is the straight segment $t = xs$, $0 \leq s \leq 1$. The integral reduces to the integral in (2.5) in the case where $\alpha = n + 1$ a positive integer so that when α is a positive integer, the integral (2.9) is a repeated indefinite integral. It is called the Riemann-Liouville fractional integral of order α. We shall denote it by the symbol

$$\mathcal{R}_\alpha\{f(t); x\}. \tag{2.10}$$

Integrals of this kind occur in the solution of ordinary linear differential equations [cf. Ince (1927), p. 191 et seq.] where they are called Euler transforms of the first kind.

There are alternative notations for $\mathcal{R}_\alpha\{f(t); x\}$ such as $I^\alpha f(x)$ used by Marcel Riess (1949) and $I^\alpha_+ f(x)$ used by other writers. Hardy and Littlewood (1928) consider the fractional integral

$$f_\alpha(x) = \int_{-\infty}^{x} f(t)(x-t)^{\alpha-1}dt, \quad (0 \leq \alpha \leq 1)$$

Love and Young (1938) consider the integral

$$f^+_\alpha(a,x) = \frac{1}{\Gamma(\alpha)}\int_a^x f(t)(x-t)^{\alpha-1}dt,$$

in which $a \leq x \leq b$, $f(x)$ being integrable in (a, b), and $\mathrm{Re}\,\alpha > 0$. Zygmund (1959) discusses the same integral but denotes it by $F_\alpha(x)$. It is easily seen that

$$f^+_\alpha(a,x) = \mathcal{R}_\alpha\{f; x\} - \mathcal{R}_\alpha\{f; a\}.$$

The only extensive table of Riemann-Liouville fractional integrals to be published is that included in vol. II of Erdelyi (1954)(pp. 185-200).

It is of interest to note the connection of fractional integrals with other integral transforms. If we denote the Laplace transform of a function $\varphi(t)$ by $\mathcal{L}\{\varphi(t); p\}$, then

$$\mathcal{L}\{\mathcal{R}_\alpha(f;t);p\} = \frac{1}{\Gamma(\alpha)}\int_0^\infty e^{-pt}dt\int_0^t f(\tau)(t-\tau)^{\alpha-1}d\tau$$

which becomes, as a result of an interchange, in the orders of the integrations

$$\frac{1}{\Gamma(\alpha)}\int_0^\infty f(\tau)d\tau\int_\tau^\infty (t-\tau)^{\alpha-1}e^{-pt}dt = p^{-\alpha}\mathcal{L}\{f(t);p\}$$

from which it follows that

$$\mathcal{L}\{\mathcal{R}_\alpha(f;t);p\} = p^{-\alpha}\mathcal{L}\{f(t);p\}. \tag{2.11}$$

If we let $\alpha \to 0$ we find that $\mathcal{L}\{R_0(f; t);p\} = \mathcal{L}\{f(t);p\}$ suggesting that we adopt the convention

$$R_0\{f(t);x\} = f(x) \tag{2.12}$$

Similarly, if we denote the Mellin transform of a function $\varphi(t)$ by $\mathcal{M}\{\varphi(t);s\}$, then

$$\mathcal{M}\{\mathcal{R}_\alpha(f;t);s\} = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{s-1}dt \int_0^t (t-\tau)^{\alpha-1}f(\tau)d\tau$$

$$= \frac{1}{\Gamma(\alpha)} \int_0^\infty f(\tau)d\tau \int_\tau^\infty t^{s-1}(t-\tau)^{\alpha-1}dt$$

$$= \frac{1}{\Gamma(\alpha)} \int_0^\infty f(\tau)d\tau \frac{\Gamma(\alpha)\Gamma(1-\alpha-s)}{\Gamma(1-s)} \tau^{\alpha+s-1}$$

from which we deduce that

$$\mathcal{M}\{\mathcal{R}_\alpha(f;t);s\} = \frac{\Gamma(1-\alpha-s)}{\Gamma(1-s)} \mathcal{M}\{f(t);s+\alpha\}. \tag{2.13}$$

The relation (2.11) can be used to derive the solution of Abel's integral equation

$$\int_0^x (x-t)^{-\alpha} f(t)dt = g(x) \tag{2.14}$$

[cf. Doetsch, (1937), p. 293, et seq.] If we take the Laplace transform of both sides of this equation we obtain the relation

$$\Gamma(1-\alpha)\mathcal{L}\{\mathcal{R}_{1-\alpha}(f;x);p\} = \bar{g}(p)$$

where $\bar{g}(p)$ denotes $\mathcal{L}\{g(x);p\}$. Making use of the result (2.11) we see that the Laplace transform of $f(x)$ is given by the equation

$$\bar{f}(p) = \frac{1}{\Gamma(1-\alpha)} p^{1-\alpha} \bar{g}(p)$$

which shows that $f(x) = F'(x)$ where

$$\bar{F}(p) = \frac{1}{\Gamma(1-\alpha)} p^{-\alpha} \bar{g}(p).$$

Using (2.11) again with the relation of $\Gamma(\alpha)\,\Gamma(1-\alpha) = \pi/\sin(\pi\alpha)$ we find

$$F(x) = \frac{\pi}{\sin(\pi\alpha)} \int_0^x \frac{g(t)dt}{(x-t)^{1-\alpha}}$$

so that the solution of the integral equation (2.14) is

$$f(x) = \frac{\pi}{\sin(\pi\alpha)} \frac{d}{dx} \int_0^x \frac{g(t)dt}{(x-t)^{1-\alpha}}. \tag{2.15}$$

By a simple change of variable we can write this result in the form: The solution of the integral equation

$$f(x) = \int_a^x \frac{g(t)dt}{(x^2-t^2)^{\mu}}, \; 0 < \mu < 1, \qquad (2.16a)$$

is

$$g(x) = \frac{2\sin(\pi\mu)}{\pi} \frac{d}{dx} \int_a^x \frac{tf(t)dt}{(x^2-t^2)^{1-\mu}}. \qquad (2.16b)$$

We shall also make use of the fact that the solution of the integral equation

$$f(x) = \int_x^b \frac{g(t)dt}{(t^2-x^2)^{\mu}}, \quad 0 < \mu < 1, \qquad (2.17a)$$

is

$$g(x) = -\frac{2\sin(\pi\mu)}{\pi} \frac{d}{dx} \int_x^b \frac{tf(t)dt}{(t^2-x^2)^{1-\mu}}, \; 0 < \mu < 1. \quad (2.17b)$$

2.3. The Weyl Fractional Integral.

The Weyl fractional integral is a generalization of the integral on the right-hand side of equation (2.8). The Weyl fractional integral of order α is defined by the equation

$$\mathfrak{W}_{\alpha} \left\{ f(t);x \right\} = \frac{1}{\Gamma(\alpha)} \int_x^{\infty} f(t)(t-x)^{\alpha-1}dt. \quad (2.18)$$

In general x and α are complex, the path of integration being one of the rays $t = xs$, $s > 0$ or $t = x + s$, $s > 0$. When they occur in the theory of linear ordinary differential equations, fractional integrals of this kind are called Euler transforms of the second kind.

A fractional integral closely related to Weyl's has been introduced by Love and Young (1938) who consider the integral

$$f_{\alpha}^{-}(x,b) = \frac{1}{\Gamma(\alpha)} \int_x^b f(t)(t-x)^{\alpha-1}dt; \qquad (2.19)$$

this integral is sometimes also denoted by $I_{-}^{\alpha}f$. Its relation to the Weyl fractional integral is expressed through the equation

$$f_{\alpha}^{-}(x,b) = \mathfrak{W}_{\alpha} \left\{ f(t)H(b-t);x \right\}. \qquad (2.20)$$

It can also be expressed in terms of the Riemann-Liouville fractional integral through the equation

$$f_\alpha^-(x,b) = \mathcal{R}_\alpha \left\{ f(b-t);\ a - x \right\} \tag{2.21}$$

Some writers also use the notations $K^\alpha f(x)$, $K_-^\alpha f(x)$ to denote $\mathfrak{W}_\alpha \{f(t);x\}$.

The only table of Weyl fractional integrals appears to be that given by Erdelyi (1954), Vol. II, pp. 201-212.

Corresponding to equation (2.11) we have

$$\begin{aligned}
\mathcal{F}\left\{ \mathfrak{W}_\alpha(f;x);\ \xi \right\} &= \frac{1}{2\pi\,\Gamma(\alpha)} \int_{-\infty}^{\infty} e^{i\xi x}\,dx \int_x^{\infty} f(t)(t-x)^{\alpha-1}dt \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)dt \cdot \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{t} e^{i\xi x}\,(t-x)^{\alpha-1}dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)e^{i\xi t}\,dt\, \frac{1}{\Gamma(\alpha)} \int_0^{\infty} e^{-i\xi\eta}\eta^{\alpha-1}d\eta \\
&= e^{-\frac{1}{2}\pi i\alpha}\, \xi^{-\alpha} \left(\frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} f(t)e^{i\xi t}\,dt \right),
\end{aligned}$$

showing that the exponential Fourier transform of a Weyl fractional integral is given by the equation

$$\mathcal{F}\left\{ \mathfrak{W}_\alpha(f;x);\ \xi \right\} = e^{-\frac{1}{2}\pi i\alpha}\, \xi^{-\alpha}\, \mathcal{F}\{f(x);\ \xi\} \tag{2.22}$$

Letting $\alpha \to 0$ in this equation we find that $\mathcal{F}\left\{\mathfrak{W}_0(f;x);\ \xi\right\} = \mathcal{F}\left\{f(x);\ \xi\right\}$ suggesting that we adopt the convention

$$\mathfrak{W}_0\left\{f(t);x\right\} = f(x).$$

3. The Erdelyi-Kober Operators.

In this section we shall consider the properties of a pair of operators which are so closely related to operators discussed by A. Erdelyi and H. Kober that it seems appropriate to call them Erdelyi-Kober operators.

3.1 Definitions:

The operators $I_{\eta,\alpha}$ and $K_{\eta,\alpha}$ are defined by the formulae

$$I_{\eta,\alpha} f(x) = \frac{2x^{-2\alpha-2\eta}}{\Gamma(\alpha)} \int_0^x (x^2 - u^2)^{\alpha-1} u^{2\eta+1} f(u)du, \tag{3.1}$$

$$K_{\eta,\alpha} f(x) = \frac{2x^{2\eta}}{\Gamma(\alpha)} \int_x^\infty (u^2 - x^2)^{\alpha-1} u^{-2\alpha-2\eta+1} f(u)du, \tag{3.2}$$

if Re $\alpha \geqslant 0$ and Re $\eta > -\frac{1}{2}$.

It is possible to extend simply the results of Erdelyi (1940) to investigate the class of functions to which $I_{\eta,\alpha}f$ belongs if $f(x)$ belongs to L_p, but we shall not discuss this problem here.

It should be noted that so far we are restricted to α's for which Re $\alpha > 0$. We shall also consider only real values of α and η and return later to the problem of defining the operators when $\alpha \geq 0$.

These operators are closely related to the Riemann-Liouville and the Weyl fractional integrals introduced in § 2. For instance it follows immediately from the definition (2.9) that

$$\begin{aligned}\mathcal{R}_\alpha \left\{t^\eta f(t^{\frac{1}{2}}); x^2\right\} &= \frac{1}{\Gamma(\alpha)} \int_0^{x^2} t^\eta f(t^{\frac{1}{2}})(x^2 - t^2)^{\alpha-1} dt \\ &= \frac{2}{\Gamma(\alpha)} \int_0^x u^{2\eta+1} f(u)(x^2 - u^2)^{\alpha-1} du \\ &= x^{2\alpha+2\eta} I_{\eta,\alpha} f\end{aligned}$$

In this way we establish the relation

$$I_{\eta,\alpha} f(x) = x^{-2\alpha-2\eta} \mathcal{R}_\alpha \left\{t^\eta f(t^{\frac{1}{2}}); x^2\right\} \tag{3.3}$$

Similarly we have the relation

$$\begin{aligned}\mathfrak{W}_\alpha \left\{t^{-\eta-\alpha} f(t^{\frac{1}{2}}); x^2\right\} &= \frac{1}{\Gamma(\alpha)} \int_{x^2}^\infty t^{-\eta-\alpha} f(t^{\frac{1}{2}})(t - x^2)^{\alpha-1} dt \\ &= \frac{2}{\Gamma(\alpha)} \int_x^\infty u^{-2\eta-2\alpha+1} f(u)(u^2 - x^2)^{\alpha-1} du\end{aligned}$$

showing that

$$K_{\eta,\alpha} f(x) = x^{2\eta} \mathfrak{W}_\alpha \left\{t^{-\eta-\alpha} f(t^{\frac{1}{2}}); x^2\right\}. \tag{3.4}$$

Some particular cases are of interest. If we let $\alpha \to 0$ and make use of the equations (2.12), (2.23) we see that the convention

$$I_{\eta,0}f(x) = f(x), \quad K_{\eta,0}f(x) = f(x) \tag{3.5}$$

is merely a restatement of the convention

$$\mathcal{R}_0\ \{f(t);\ x\} = f(x), \quad \mathcal{W}_0\ \{f(t);\ x\} = f(x).$$

Further, if we put $\eta = 0$ in equations (3.3) and (3.4) we obtain the relations

$$I_{0,\alpha}f(x) = x^{-2\alpha}\mathcal{R}_\alpha\ \{f(t^{\frac{1}{2}});\ x^2\}, \quad K_{0,\alpha}f(x) = \mathcal{W}_\alpha\ \{t^{-\alpha}f(t^{\frac{1}{2}});\ x^2\}. \tag{3.6}$$

Because of the relations (3.3), (3.4) it is a simple matter to use the tables in Erdelyi (1954) to calculate $I_{\eta,\alpha}f(x)$ and $K_{\eta,\alpha}f(x)$ for any prescribed function $f(x)$. For example, if

$$f(x) = x^{2\beta}(x^2 + c^2)^\gamma,$$

then

$$t^\eta f(t^{\frac{1}{2}}) = t^{\beta+\eta}(t + c^2)^\gamma$$

Now, from entry (9) on p. 186 of Vol. II of Erdelyi (1954), we know that if $\alpha > 0$, $\beta + \eta + 1 > 0$, then

$$\mathcal{R}_\alpha\left\{t^{\beta+\eta}(t+c^2)^\gamma;x\right\} = \frac{c^{2\gamma}x^{\alpha+\beta+\eta}\,\Gamma(\beta+\eta+1)}{\Gamma(\alpha+\beta+\eta+1)}\ {}_2F_1(-\gamma,\beta+\eta+1;\ \alpha+\beta+\eta+1;\ -\frac{x}{c^2})$$

if $|\arg x/c^2| < \pi$. It follows immediately from equation (3.3) that for this $f(x)$

$$I_{\eta,\alpha}f(x) = \frac{c^{2\gamma}x^{2\beta}\ \Gamma(\beta+\eta+1)}{\Gamma(\alpha+\beta+\eta+1)}\ {}_2F_1(-\gamma,\ \beta+\eta+1;\ \alpha+\beta+\eta+1;\ -\frac{x^2}{c^2})$$

if $|\arg x/c| < \frac{1}{2}\pi$.

It is often a simple matter to calculate $I_{\eta,\alpha}f$ directly. For example, since

$$\int_0^x (x^2 - u^2)^{\alpha-1}\,u^{2\eta+1+2r}\,du = \frac{1}{2}\Gamma(\alpha)\frac{\Gamma(\eta+1+r)}{\Gamma(\alpha+\eta+1+r)}\,x^{2\alpha+2\eta+2r}$$

it follows that

$$I_{\eta,\alpha}\ {}_pF_q(a_1,\ \ldots,\ a_p;\ b_1,\ \ldots,\ b_q;\ u^2)$$

$$= \frac{\Gamma(\eta+1)}{\Gamma(\alpha+\eta+1)}\,{}_{p+1}F_{q+1}(a_1,\ldots,a_p,\ \eta+1;\ b_1,\ldots,b_q,\alpha+\eta+1;\ x^2) \tag{3.7}$$

where ${}_pF_q$ denotes a generalised hypergeometric function.

3.2 Properties of the operators.

We shall now derive the basic properties of the operators $I_{\eta,\alpha}$ and $K_{\eta,\alpha}$, assuming for the moment that $\alpha \geq 0$. (We cannot do otherwise since we have not yet defined the operators for negative values of α).

Since

$$I_{\eta,\alpha}x^{2\beta}f(x) = \frac{2x^{-2\alpha-2\eta}}{\Gamma(\alpha)}\int_0^x (x^2-u^2)^{\alpha-1}u^{2\eta+1}u^{2\beta}f(u)du$$

$$= x^{2\beta}\frac{2x^{-2\alpha-2\beta-2\eta}}{\Gamma(\alpha)}\int_0^x (x^2-u^2)^{\alpha-1}u^{2(\eta+\beta)+1}f(u)du$$

we have the relation

$$I_{\eta,\alpha}x^{2\beta}f(x) = x^{2\beta}I_{\eta+\beta,\alpha}f(x). \tag{3.8}$$

If we write down the expressions for $I_{\eta,\alpha}$ and $I_{\eta+\alpha,\beta}f$ we find that

$$I_{\eta,\alpha}I_{\eta+\alpha,\beta}f$$

$$= \frac{2x^{-2\eta-2\alpha}}{\Gamma(\alpha)}\int_0^x (x^2-u^2)^{\alpha-1}u^{2\eta+1}du\,\frac{2u^{-2\eta-2\alpha-2\beta}}{\Gamma(\beta)}\int_0^u (u^2-v^2)^{\beta-1}v^{2\eta+2\alpha+1}f(v)dv$$

Interchanging the order of the integrations and evaluating the inner integral by means of equation (1.1) we find that

$$I_{\eta,\alpha}I_{\eta+\alpha,\beta}f = \frac{2x^{-2\eta-2\alpha-2\beta}}{\Gamma(\alpha+\beta)}\int_0^x v^{2\eta+1}(x^2-v^2)^{\alpha+\beta-1}f(v)dv = I_{\eta,\alpha+\beta}f$$

so that we have the product rule

$$I_{\eta,\alpha}I_{\eta+\alpha,\beta} = I_{\eta,\alpha+\beta} \tag{3.9}$$

There are corresponding rules for $K_{\eta,\alpha}$. Since

$$K_{\eta,\alpha}x^{2\beta}f(x) = x^{2\beta}\frac{2x^{2(\eta-\beta)}}{\Gamma(\alpha)}\int_x^\infty (u^2-x^2)^{\alpha-1}u^{-2(\eta-\beta)-2\alpha+1}f(u)du$$

we see that

$$K_{\eta,\alpha}x^{2\beta}f(x) = x^{2\beta}K_{\eta+\beta,\alpha}f(x). \tag{3.10}$$

Also, since

$K_{\eta,\alpha} K_{\eta+\alpha,\beta} f(x)$

$$= \frac{4x^{2\eta}}{\Gamma(\alpha)\Gamma(\beta)} \int_x^\infty (u^2 - x^2)^{\alpha-1} u du \int_u^\infty (v^2 - u^2)^{\beta-1} v^{-2\eta-2\alpha-2\beta+1}$$

we find, on interchanging the order of the integrations and making use of equation (1.2), that this repeated integral is equal to

$$\frac{2x^{2\eta}}{\Gamma(\alpha+\beta)} \int_x^\infty (v^2 - x^2)^{\alpha+\beta-1} v^{-2\eta-2\alpha-2\beta+1} f(v) dv$$

showing that

$$K_{\eta,\alpha} K_{\eta+\alpha,\beta} = K_{\eta,\alpha+\beta}. \tag{3.11}$$

3.3 Definitions for negative α.

The results we have just established suggest the manner in which we should define the operators $I_{\eta,\alpha}$ and $K_{\eta,\alpha}$ for $\alpha < 0$.

From the results (3.9) and (3.5) we have formally

$$I_{\eta+\alpha,-\alpha} I_{\eta,\alpha} f = I_{\eta,0} f = f. \tag{3.12}$$

This suggests that, if $\alpha < 0$, we define

$$g = I_{\eta,\alpha} f \tag{3.13a}$$

to be the solution of the integral equation

$$I_{\eta+\alpha,-\alpha} g = f. \tag{3.13b}$$

Similarly the equations (3.11) and (3.5) suggest that, if $\alpha < 0$, we define

$$h = K_{\eta,\alpha} f \tag{3.14a}$$

to be the solution of the integral equation

$$K_{\eta+\alpha,-\alpha} h = f. \tag{3.14b}$$

The equation (3.12) can also be used to give us the inverse of the $I_{\eta,\alpha}$ operator; we have

$$I^{-1}_{\eta,\alpha} = I_{\eta+\alpha,-\alpha} \tag{3.15}$$

Similarly, we have the relation

$$K^{-1}_{\eta,\alpha} = K_{\eta+\alpha,-\alpha} \tag{3.16}$$

3.4 The operator $\mathcal{D}_x$.

We now introduce the operator

$$\mathcal{D}_x = \tfrac{1}{2}\frac{d}{dx}x^{-1} \tag{3.17}$$

to obtain relations from which we derive formulae by means of which we may calculate $I_{\eta,\alpha}$ and $K_{\eta,\alpha}$ when $\alpha < 0$.

By the definition of the $I_{\eta,\alpha}$ operator we have that

$$I_{\eta,\alpha}\{t^{-2\eta-1}\mathcal{D}_t^n\, t^{2n+\nu+1} f(t);\, x\}$$

$$= \frac{2x^{-2\alpha-2\eta}}{\Gamma(\alpha)}\int_0^x (x^2-t^2)^{\alpha-1}\mathcal{D}_t^n\, t^{2n+\nu+1} f(t)dt.$$

Applying the formula for integrating by parts we find that the integral on the right-hand side of this equation becomes

$$\frac{2x^{-2\alpha-2\eta}}{\Gamma(\alpha)}\{[(x^2-t^2)^{\alpha-1}\tfrac{1}{2}t^{-1}\mathcal{D}_t^{n-1}\, t^{2n+\nu+1} f(t)]_0^x$$

$$+ (\alpha-1)\int_0^x (x^2-t^2)^{\alpha-2}\mathcal{D}_t^{n-1}\, t^{2n+\nu+1} f(t)dt\}$$

$$= x^{-2}\,\frac{2x^{-2(\alpha-1)-2\eta}}{\Gamma(\alpha-1)}\int_0^x (x^2-t^2)^{\alpha-2}\mathcal{D}_t^{n-1}\, t^{2n+\nu+1} f(t)dt$$

showing that

$$I_{\eta,\alpha}\{t^{-2\eta-1}\mathcal{D}_t^n\, t^{2n+\nu+1} f(t);x\} = x^{-2}\, I_{\eta,\alpha-1}\{t^{-2\eta-1}\mathcal{D}_t^{n-1}\, t^{2n+\nu+1} f(t);x\}.$$

If we repeat the process $\underline{n}$ times we obtain the relation

$$I_{\eta,\alpha}\{t^{-2\eta-1}\mathcal{D}_t^n\, t^{2n+\nu+1} f(t);x\} = x^{-2n}\, I_{\eta,\alpha-n}\{t^{2n-2\eta+\nu} f(t);x\}. \tag{3.18}$$

Making the substitution $\alpha = n$, replacing η by $\eta - n$, and making use of equation (3.5) we find that this result can be written in the form

$$I_{\eta-n,n}\{t^{-2\eta+2n-1}\mathcal{D}_t^n\, t^{2n+\nu+1} f(t);x\} = x^{+2n-2\eta+\nu}f(x)$$

and using the relation (3.15) we have the relation

$$I_{\eta,-n}\, x^{+2n-2\eta+\nu} f(x) = x^{2n-2\eta-1}\mathcal{D}_x^n\, x^{2n+\nu+1} f(x). \tag{3.19}$$

Putting $\nu = 2\eta - 2n$, we obtain the important particular case

$$I_{\eta,-n}\, f(x) = x^{2n-2\eta-1} \mathfrak{D}_x^n\, x^{2\eta+1} f(x), \quad n > 0. \tag{3.20}$$

If we replace η by $\eta + \alpha$, α by $-\alpha$ and ν by $2\alpha + 2\eta$ in equations (3.18) we find that

$$I_{\eta+\alpha,-\alpha} \{t^{-2\eta-2\alpha-1} \mathfrak{D}_t^n\, t^{2n+2\alpha+2\eta+1}\, I_{\eta,\alpha+n}\, f(t)\}$$

$$= x^{-2n}\, I_{\eta+\alpha,-\alpha-n} \left\{t^{2n}\, I_{\eta,\alpha+n}\, f(t)\right\}.$$

Using equation (3.8) we see that the right-hand side of this equation is equal to $I_{\eta+\alpha+n,-\alpha-n}\, I_{\eta,\alpha+n}\, f(t)$, which, by equation (3.12) is equal to $f(x)$. Hence we find that

$$I^{-1}_{\eta+\alpha,-\alpha}\, f(x) = x^{-2\eta-2\alpha-1} \mathfrak{D}_x^n\, x^{2n+2\alpha+2\eta+1}\, I_{\eta,\alpha+n}\, f(x)$$

which, as a consequence of equation (3.13) can be written in the form

$$I_{\eta,\alpha}\, f(x) = x^{-2\eta-2\alpha-1} \mathfrak{D}_x^n\, x^{2n+2\alpha+2\eta+1}\, I_{\eta,\alpha+n}\, f(x). \tag{3.21}$$

The special case in which $-1 < \alpha < 0$ is of some importance. Then we take $n = 1$ and obtain the expression

$$I_{\eta,\alpha}\, f(x) = \tfrac{1}{2}\, x^{-2\eta-2\alpha-1} \frac{d}{dx}\, x^{2\alpha+2\eta+2}\, I_{\eta,\alpha+1}\, f(x), \quad 0 > \alpha > -1$$

Inserting the expression for $I_{\eta,1+\alpha} f(x)$ we find that

$$I_{\eta,\alpha} f(x) = \frac{x^{-2\eta-2\alpha-1}}{\Gamma(1+\alpha)} \frac{d}{dx} \int_0^x (u^2 - x^2)^\alpha\, u^{2\eta+1} f(u)\,du, \quad -1 < \alpha < 0. \tag{3.22}$$

We have similar results for the operator $K_{\eta,\alpha}$. For example, corresponding to equation (3.18) we have the result

$$K_{\eta,\alpha} \{x^{2\alpha+2\eta-1} \mathfrak{D}_x^n\, x^\nu f(x)\} = (-1)^n\, K_{\eta,\alpha-n} \{x^{2\alpha-2n+2\eta+\nu-1} f(x)\} \tag{3.23}$$

Replacing η by $\eta - n$, α by n and ν by $2n - 2\eta + 1$ we find that

$$K_{\eta-n,n} \{x^{2\eta-1} \mathfrak{D}_x^n\, x^{2n-2\eta+1} f(x)\} = (-1)^n f(x),$$

a result which, because of equation (3.16) may be written in the form

$$K_{\eta,-n} f(x) = (-1)^n x^{2\eta-1} \mathcal{D}_x^n x^{2n-2\eta+1} f(x), \quad n > 0. \tag{3.24}$$

Finally we have the formula

$$K_{\eta,\alpha} f(x) = (-1)^n x^{2\eta-1} \mathcal{D}_x^n x^{2n-2\eta+1} K_{\eta-n,\alpha+n} f(x), \; n > 0, \tag{3.25}$$

which can be used to evaluate $K_{\eta,\alpha}f$ when $\alpha < 0$ and n is a positive integer such that $0 \leq \alpha + n < 1$.

3.5 Fractional integration by parts.

This is the name given to the formula

$$\int_0^\infty xf(x)I_{\eta,\alpha}\, g(x)dx = \int_0^\infty xg(x)K_{\eta,\alpha}\, f(x)dx \tag{3.26}$$

derived by Erdelyi (1940).

The proof is straightforward. We have

$$\int_0^\infty xf(x)I_{\eta,\alpha}\, g(x)dx = \frac{2}{\Gamma(\alpha)}\int_0^\infty x^{1-2\eta-2\alpha} f(x)dx \int_0^x (x^2-t^2)^{\alpha-1} t^{2\eta+1} g(t)dt$$

$$= \frac{2}{\Gamma(\alpha)}\int_0^\infty t^{2\eta+1}\, g(t)dt \int_t^\infty (x^2-t^2)^{\alpha-1} x^{1-2\eta-2\alpha} f(x)dx$$

$$= \int_0^\infty tg(t)K_{\eta,\alpha}f(t)dt,$$

proving the result.

4. Modified Operator of Hankel Transforms.

In this section we shall define a modified operator of Hankel transforms closely related to that introduced by Erdelyi and Kober and discuss some of its properties.

4.1 Definition and Inversion Theorem.

The Hankel transform $\mathcal{H}_\nu\{f(t);\, x\}$ of order ν of a function f is defined by the equation

$$F(x) = \mathcal{H}_\nu\{f(t);\, x\} = \int_0^\infty tf(t)J_\nu(xt)dt. \tag{4.1}$$

By the Hankel inversion theorem we have that

$$f(t) = \mathfrak{H}_\nu\{\bar{f}(x); t\}. \tag{4.2}$$

We define the modified operator of Hankel transforms, $S_{\eta,\alpha}$, by the formula

$$S_{\eta,\alpha} f(x) = 2^\alpha x^{-\alpha} \int_0^\infty t^{1-\alpha} f(t) J_{2\eta+\alpha}(xt)\,dt, \tag{4.3}$$

so that*

$$S_{\eta,\alpha} f(x) = 2^\alpha x^{-\alpha} \mathfrak{H}_{2\eta+\alpha}\{t^{-\alpha} f(t); x\} \tag{4.4}$$

Applying the Hankel inversion theorem we find that

$$t^{-\alpha} f(t) = 2^{-\alpha} \mathfrak{H}_{2\eta+\alpha}\{x^\alpha S_{\eta,\alpha} f(x); t\}$$

so that

$$f(t) = S_{\eta+\alpha,-\alpha}\{S_{\eta,\alpha} f(x); t\}$$

from which it follows that

$$S_{\eta+\alpha,-\alpha} S_{\eta,\alpha} = I,$$

where I is the identity operator. Hence the inversion theorem for the modified operator of Hankel transforms can be written in the form

$$S_{\eta,\alpha}^{-1} = S_{\eta+\alpha,-\alpha} \tag{4.5}$$

4.2 Relations between the modified operator of Hankel transforms and the Erdelyi-Kober operators.

Between the Erdelyi-Kober operators of fractional integration on the one hand and operators of Hankel transforms on the other we have a set of interesting relations.

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*In calculations involving these operators it is sometimes convenient to use equation (4.4) in the form $\mathfrak{H}_p\{t^q f(t); x\} = 2^q x^{-q} S_{\frac{1}{2}p+\frac{1}{2}q,-q} f(x)$.

For instance, from the definitions of the operators $I_{\eta+\alpha,\beta}$ and $S_{\eta,\alpha}$ we find that

$$I_{\eta+\alpha,\beta}\, S_{\eta,\alpha} f(x)$$

$$= \frac{2x^{-2\alpha-2\beta-2\eta}}{\Gamma(\beta)} \int_0^x (x^2 - u^2)^{\beta-1}\, u^{2\alpha+2\eta+1}\, 2^{\alpha}\, u^{-\alpha} \int_0^{\infty} y^{1-\alpha} f(y) J_{2\eta+\alpha}(uy)\,dy\,du.$$

Interchanging the order of the integrations we find that this integral is equal to

$$\frac{2^{\alpha+1}x^{-2\alpha-2\beta-2\eta}}{\Gamma(\beta)} \int_0^{\infty} y^{1-\alpha} f(y)dy \int_0^x (x^2 - u^2)^{\beta-1}\, u^{\alpha+2\eta+1}\, J_{2\eta+\alpha}(uy)du$$

The inner integral is Sonine's first integral (1.4) so that the repeated integral reduces to the single integral

$$2^{\alpha+\beta}\, x^{-\alpha-\beta} \int_0^{\infty} y^{1-\alpha-\beta} f(y) J_{2\eta+\alpha+\beta}(xy)dy$$

which is merely $S_{\eta,\alpha+\beta} f(x)$. In this way we have established the relation

$$I_{\eta+\alpha,\beta}\, S_{\eta,\alpha} = S_{\eta,\alpha+\beta} \tag{4.6}$$

In a similar way we have

$$K_{\eta,\alpha} S_{\eta+\alpha,\beta} f(x)$$

$$= \frac{2^{\beta+1}\, x^{2\eta}}{\Gamma(\alpha)} \int_x^{\infty} (u^2 - x^2)^{\alpha-1}\, u^{-2\alpha-2\eta+1} du\, u^{-\beta} \int_0^{\infty} y^{1-\beta} f(y) J_{2\eta+2\alpha+\beta}(uy)dy$$

$$= \frac{2^{\beta+1}\, x^{2\eta}}{\Gamma(\alpha)} \int_0^{\infty} y^{1-\beta} f(y)dy \int_x^{\infty} (u^2 - x^2)^{\alpha-1}\, u^{-2\alpha-\beta-2\eta+1} J_{2\eta+2\alpha+\beta}(uy)du$$

The inner integral can be evaluated by means of the formula (1.6) and we find that the repeated integral has the value

$$2^{\alpha+\beta}\, x^{-\alpha-\beta} \int_0^{\infty} y^{1-\alpha-\beta} f(y) J_{2\eta+\alpha+\beta}(xy)dy = S_{\eta,\alpha+\beta} f(x),$$

showing that

$$K_{\eta,\alpha}\, S_{\eta+\alpha,\beta} = S_{\eta,\alpha+\beta}. \tag{4.7}$$

Further,

$S_{\eta+\alpha,\beta}\, S_{\eta,\alpha} f(x)$

$$= 2^{\alpha+\beta} x^{-\beta} \int_0^\infty y^{1-\beta} J_{2\eta+2\alpha+\beta}(xy)dy \cdot y^{-\alpha} \int_0^\infty u^{1-\alpha} J_{2\eta+\alpha}(yu) f(u)du$$

$$= 2^{\alpha+\beta} x^{-\beta} \int_0^\infty u^{1-\alpha} f(u)du \int_0^\infty y^{1-\alpha-\beta} J_{2\eta+\alpha}(yu) J_{2\eta+2\alpha+\beta}(xy)dy.$$

The inner integral can be evaluated by means of equation (1.3) and we find that the repeated integral is equal to

$$\frac{2}{\Gamma(\alpha+\beta)} x^{-2\alpha-2\beta-2\eta} \int_0^x u^{1+2\eta} (x^2 - u^2)^{\alpha+\beta-1} f(u)du,$$

that is, to $I_{\eta,\alpha+\beta} f(x)$. Hence we have shown that

$$S_{\eta+\alpha,\beta}\, S_{\eta,\alpha} = I_{\eta,\alpha+\beta}. \tag{4.8}$$

There is a similar formula for the product $S_{\eta,\alpha}\, S_{\eta+\alpha,\beta}$. In this case we have

$S_{\eta,\alpha}\, S_{\eta+\alpha,\beta} f(x)$

$$= 2^{\alpha+\beta} x^{-\alpha} \int_0^\infty y^{1-\alpha} J_{2\eta+\alpha}(xy)dy \cdot y^{-\beta} \int_0^\infty u^{1-\beta} J_{2\eta+2\alpha+\beta}(uy) f(u)du$$

$$= 2^{\alpha+\beta} x^{-\alpha} \int_0^\infty u^{1-\beta} f(u)du \int_0^\infty y^{1-\alpha-\beta} J_{2\eta+\alpha}(xy) J_{2\eta+2\alpha+\beta}(uy)dy.$$

Again using equation (1.3), but noting that here the roles of x and u are interchanged, we evaluate the inner integral to get

$$\frac{2x^{\eta}}{(\alpha+\beta)} \int_x^\infty u^{1-2\alpha-2\beta-2\eta} (u^2 - x^2)^{\alpha+\beta-1} f(u)du = K_{\eta,\alpha+\beta} f(x)$$

for the repeated integral. In other words we have shown that

$$S_{\eta,\alpha}\, S_{\eta+\alpha,\beta} = K_{\eta,\alpha+\beta} \tag{4.9}$$

From the definition of the operators involved we see that

$S_{\alpha+\eta,\beta}\, I_{\eta,\alpha} f(x)$

$$= 2^{\beta} x^{-\beta} \int_0^\infty t^{1-\beta} J_{2\alpha+\beta+2\eta}(xt)dt \,.\, \frac{2t^{-2\alpha-2\eta}}{\Gamma(\alpha)} \int_0^t (t^2 - u^2)^{\alpha-1} u^{2\eta+1} f(u)du$$

$$= \frac{2^{\beta+1} x^{-\beta}}{\Gamma(\alpha)} \int_0^\infty u^{2\eta+1} f(u)du \int_u^\infty t^{1-2\alpha-\beta-2\eta} (t^2 - u^2)^{\alpha-1} J_{2\alpha+\beta+2\eta}(xt)dt$$

$$= 2^{\alpha+\beta} x^{-\alpha-\beta} \int_0^\infty u^{1-\alpha-\beta} J_{2\eta+\alpha+\beta}(xu) f(u)du.$$

on evaluating the inner integral by means of equation (1.6). This last integral is $S_{\eta,\alpha+\beta} f(x)$ so that we have shown that

$$S_{\eta+\alpha,\beta}\, I_{\eta,\alpha} = S_{\eta,\alpha+\beta}. \tag{4.10}$$

By a similar process we can show that

$$S_{\eta,\alpha}\, K_{\eta+\alpha,\beta} = S_{\eta,\alpha+\beta}. \tag{4.11}$$

Written in terms of the operator of Hankel transforms $\mathcal{H}_\nu$ these last two equations take the forms

$$\mathcal{H}_{2\eta+2\alpha+\beta}\{x^{-\beta} I_{\eta,\alpha} f(x);\xi\} = 2^{\alpha} \xi^{-\alpha} \mathcal{H}_{2\eta+\alpha+\beta}\{x^{-\alpha-\beta} f(x);\xi\} \tag{4.12}$$

and

$$\mathcal{H}_{2\eta+\alpha}\{x^{-\alpha} K_{\eta+\alpha,\beta} f(x);\xi\} = 2^{\beta} \xi^{-\beta} \mathcal{H}_{2\eta+\alpha+\beta}\{x^{-\alpha-\beta} f(x);\xi\} \tag{4.13}$$

respectively.

Particular cases of these results are of some interest in the solution of boundary value problems [cf. Sneddon (1962)].

If we put $\alpha = \frac{1}{2}$, $\beta = 0$, $\eta = -\frac{1}{2}$ in (4.12) and make use of the fact that since

$$\cos(\xi x) = (\tfrac{1}{2}\pi\xi x)^{\frac{1}{2}} J_{-\frac{1}{2}}(\xi x)$$

the Fourier cosine transform of a function may be written as

$$\mathcal{F}_c\{f(x);\xi\} = \xi^{\frac{1}{2}} \mathcal{H}_{-\frac{1}{2}}\{x^{-\frac{1}{2}} f(x);\xi\}\,, \tag{4.14}$$

we find that

$$\mathcal{H}_0\{g(\rho);\xi\} = \xi^{-1} \mathcal{F}_c\{f(x);\xi\}\,, \tag{4.15}$$

where

$$g(\rho) = 2^{-\frac{1}{2}} I_{-\frac{1}{2},\frac{1}{2}} f(\rho) = \sqrt{\frac{2}{\pi}} \int_0^\rho \frac{f(x)dx}{\sqrt{(\rho^2 - x^2)}} . \tag{4.16}$$

Similarly if we put $\alpha = -\frac{1}{2}$, $\beta = \frac{1}{2}$, $\eta = 0$ in (4.13) we find that

$$\mathcal{H}_0 \{f(\rho); \xi\} = \xi^{-1} \mathcal{F}_c \{ h(x); \xi\} , \tag{4.17}$$

where

$$h(x) = 2^{-\frac{1}{2}} x K_{-\frac{1}{2},\frac{1}{2}} f(x) = \sqrt{\frac{2}{\pi}} x \int_x^\infty \frac{f(\rho)d\rho}{\sqrt{(\rho^2 - x^2)}} . \tag{4.18}$$

If we put $\alpha = \frac{1}{2}$, $\beta = 0$, $\eta = 0$ in (4.12) we find that

$$\mathcal{H}_1 \{j(\rho); \xi\} = \xi^{-1} \mathcal{F}_s \{f(x), \xi\} \tag{4.19}$$

where

$$j(\rho) = 2^{-\frac{1}{2}} I_{0,\frac{1}{2}} f(\rho) = \sqrt{\frac{2}{\pi}} \int_0^\rho \frac{xf(x)dx}{\sqrt{(\rho^2 - x^2)}} , \tag{4.20}$$

and

$$\mathcal{F}_s \{f(x); \xi\} = \xi^{\frac{1}{2}} \mathcal{H}_{\frac{1}{2}} \{x^{-\frac{1}{2}} f(x); \xi\} \tag{4.21}$$

is the Fourier sine transform of $f(x)$.

Finally putting $\alpha = \beta = \frac{1}{2}$, $\eta = 0$ in (4.13) we find that

$$\mathcal{H}_1 \{f(\rho); \xi\} = \mathcal{F}_s \{ h(x); \xi\} , \tag{4.22}$$

where

$$h(x) = 2^{-\frac{1}{2}} K_{\frac{1}{2},\frac{1}{2}} f(x) = \sqrt{\frac{2}{\pi}} x \int_x^\infty \frac{f(\rho)d\rho}{\rho \sqrt{(\rho^2 - x^2)}} . \tag{4.23}$$

5. Sonine operators

We shall now introduce two operators $P^k_{\eta,\alpha}$ and $Q^k_{\eta,\alpha}$ whose two main properties we shall derive by using Sonine's second integral. For that reason we shall call them Sonine operators.

We define $P^k_{\eta,\alpha}$, $Q^k_{\eta,\alpha}$ by the equations

$$P^k_{\eta,\alpha} f = \int_0^x t^{\eta+1} I_\alpha \{ k\sqrt{(x^2 - t^2)}\} (x^2 - t^2)^{\frac{1}{2}\alpha} f(t)dt, \quad (\alpha > -1) \tag{5.1}$$

$$Q^k_{\eta,\alpha} f = \int_x^\infty t^{1-\eta} J_\alpha \{ k\sqrt{(t^2 - x^2)}\} (t^2 - x^2)^{\frac{1}{2}\alpha} f(t)dt. \quad (\alpha > -1) \tag{5.2}$$

From these definitions we have

$$P^k_{\mu,\beta}\,\mathcal{H}_\mu\{\xi^\gamma\varphi(\xi)H(\xi-k);t\}$$

$$=\int_0^x t^{\mu+1} I_\beta\{k\sqrt{(x^2-t^2)}\}(x^2-t^2)^{\frac{1}{2}\beta}\,dt\int_k^\infty \xi^{\gamma+1}\varphi(\xi)J_\mu(\xi t)d\xi$$

$$=\int_k^\infty \xi^{\gamma+1}\varphi(\xi)d\xi\int_0^x t^{\mu+1} J_\mu(\xi t)I_\beta\{k\sqrt{(x^2-t^2)}\}(x^2-t^2)^{\frac{1}{2}\beta}\,dt$$

Evaluating the inner integral by means of equation (1.9) we find that this repeated integral takes the form

$$k^\beta x^{\beta+\mu+1}\int_k^\infty \xi^{\mu+\gamma+1}\varphi(\xi)(\xi^2-k^2)^{-\frac{1}{2}\beta-\frac{1}{2}\mu-\frac{1}{2}}J_{\beta+\mu+1}\{x\sqrt{(\xi^2-k^2)}\}\,d\xi$$

If we change the variable of integration to $\tau=\sqrt{(\xi^2-k^2)}$ we find that we obtain the result

$$P^k_{\mu,\beta}\mathcal{H}_\mu\{\xi^\gamma\varphi(\xi)H(\xi-k);t\}=k^\beta x^\lambda\mathcal{H}_\lambda\{\tau^{-\lambda}(\tau^2+k^2)^{\frac{1}{2}(\mu+\gamma)}\varphi[\sqrt{(\tau^2+k^2)}];x\}\quad(5.3)$$

with

$$\lambda=\beta+\mu+1\qquad(5.4)$$

Similarly we can show that

$$Q^k_{\nu,\delta}\mathcal{H}_\nu\{\xi^{-1}(\xi^2-k^2)^\alpha\varphi(\xi);t\}=k^\delta x^{-r}\mathcal{H}_r\{\tau^{2\alpha+r}(\tau^2+k^2)^{-\frac{1}{2}\nu-\frac{1}{2}}\varphi[\sqrt{(\tau^2+k^2)}];x\}\qquad(5.5)$$

where

$$r=\delta-\nu+1\qquad(5.6)$$

We have so far defined these operators only if $\alpha>-1$. If $\alpha<-1$ we define them in the following way. Since

$$\frac{d}{du}u^{\alpha+1}J_{\alpha+1}(ku)=ku^{\alpha+1}J_\alpha(ku)$$

it is easily shown that if $\alpha>-1$

$$\frac{1}{kx}\frac{d}{dx}P^k_{\eta,\alpha+1}f=P^k_{\eta,\alpha}f$$

and if we repeat the process <u>n</u> times (where <u>n</u> is a positive integer) we have the relation

$$k^{-n}\left(\frac{1}{x}\frac{d}{dx}\right)^n P^k_{\eta,\alpha+n}f=P^k_{\eta,\alpha}f.\qquad(5.7)$$

This relation suggests the definition of $P^k_{\eta,\alpha}$ for $\alpha < 0$. We choose a positive integer $\underline{n}$ such that $\underline{n} - 1 < -\alpha < n$ and define

$$P^k_{\eta,\alpha} f = k^{-n} \left(\frac{1}{x}\frac{d}{dx}\right)^n P^k_{\eta,\alpha+n} f. \tag{5.8}$$

Similarly we are led by the relation

$$\frac{1}{kx}\frac{d}{dx} Q^k_{\eta,\alpha+1} f = -Q^k_{\eta,\alpha} f$$

to define $Q^k_{\eta,\alpha} f$ for $\alpha < -1$ by the equation

$$Q^k_{\eta,\alpha} = (-k)^{-n} \left(\frac{1}{x}\frac{d}{dx}\right)^n Q^k_{\eta,\alpha+n}, \quad (\underline{n} - 1 < -\alpha < \underline{n}). \tag{5.9}$$

6. Dual Integral Equations of Titchmarsh Type

Dual integral equations of the type

$$\int_0^\infty \xi^{-2\alpha} \varphi(\xi) J_\mu(x\,\xi)d\xi = F(x), \quad 0 < x < 1, \tag{6.1}$$

$$\int_0^\infty \xi^{-2\beta} \varphi(\xi) J_\nu(x\,\xi)d\xi = G(x), \qquad x > 1 \tag{6.2}$$

were first discussed systematically by Titchmarsh. Although Titchmarsh discussed only the case in which $\mu = \nu$ and $G(x) = 0$ it is appropriate to describe the equations (6.1) and (6.2) as being of Titchmarsh type. If we make the substitutions

$$\varphi(\xi) = \xi\psi(\xi), \quad f(x) = 2^{+2\alpha} x^{-2\alpha} F(x), \quad g(x) = 2^{+2\beta} x^{-2\beta} G(x), \tag{6.3}$$

we see that equations (6.1) and (6.2) may be written in the operator form

$$S_{\frac{1}{2}\mu-\alpha,2\alpha}\psi = f, \qquad S_{\frac{1}{2}\nu-\beta,2\beta}\psi = g, \tag{6.4}$$

where the function $f(x)$ is prescribed on the interval $I_1 = \{x:\ 0 \leq x < 1\}$ and the function $g(x)$ is prescribed on the interval $I_2 = \{x:\ x > 1\}$.

We shall find it convenient to write any function $f(x)$ defined on the positive real axis as a sum

$$f(x) = f_1(x) + f_2(x)$$

where $f_1(x)$ is defined by the equations

$$f_1(x) = \begin{cases} f(x), & x \in I_1, \\ 0, & x \in I_2, \end{cases}$$

and $f_2(x)$ by the equations

$$f_2(x) = \begin{cases} 0, & x \in I_1 \\ f(x), & x \in I_2. \end{cases}$$

In the problem we have posed we may say that $f_1(x)$ and $g_2(x)$ are known but that $f_2(x)$ and $g_1(x)$ are not known.

We shall now discuss some of the methods of solving the pair of equations (6.1) and (6.2).

6.1 Peters' Solution.

If we introduce the number

$$\lambda = \tfrac{1}{2}(\mu + \nu) - (\alpha - \beta) \tag{6.5}$$

then it follows from equation (4.6) that

$$I_{\frac{1}{2}\mu+\alpha,\, \lambda-\mu}\, S_{\frac{1}{2}\mu-\alpha, 2\alpha} = S_{\frac{1}{2}\mu-\alpha,\, \lambda-\mu+2\alpha}$$

Similarly from equation (4.7) we have the relation

$$K_{\lambda-\frac{1}{2}\nu-\beta,\, \nu-\lambda}\, S_{\frac{1}{2}\nu-\beta, 2\beta} = S_{\frac{1}{2}\mu-\alpha,\, \lambda-\mu+2\alpha}$$

It follows immediately from equations (6.4) that if we define the function $h(x)$ by the equations

$$h_1(x) = I_{\frac{1}{2}\mu+\alpha,\, \lambda-\mu} f(x), \qquad h_2(x) = K_{\frac{1}{2}\mu-\alpha,\, \nu-\lambda}\, g(x)$$

then

$$S_{\frac{1}{2}\mu-\alpha,\, \lambda-\mu+2\alpha}\, \psi = h$$

so that

$$\psi(\xi) = S^{-1}_{\frac{1}{2}\mu-\alpha,\, \lambda-\mu+2\alpha}\, h(\xi)$$

Using the inversion formula (4.5) and reverting to the original variables by means of equations (6.3) we find that the solution of the equations (6.1) and (6.2) can be written in the form

$$\varphi(\xi) = \xi\, S_{\frac{1}{2}\nu+\beta,\, \lambda-\nu-2\beta}\, h(\xi) \tag{6.6a}$$

where

$$h_1(\xi) = 2^{2\alpha}\,\xi^{-2\alpha}\, I_{\frac{1}{2}\mu,\ \lambda-\mu}F(\xi), \quad h_2(\xi) = 2^{2\beta}\,\xi^{-2\beta}\, K_{\frac{1}{2}\mu-\alpha+\beta,\ \nu-\lambda}\,G(\xi) \tag{6.6b}$$

Peters' solution [Peters (1961)] corresponds, in our notation, to the values $\alpha = -\frac{1}{2}\omega$, $\beta = 0$ in which case

$$\lambda = \tfrac{1}{2}(\mu + \nu + \omega). \tag{6.7}$$

In this case equation (6.6a) can be written in the form

$$\varphi(\xi) = 2^{\lambda-\nu}\,\xi^{1-\lambda+\nu}\left\{\int_0^1 t^{1-\lambda+\nu}\,h_1(t)J_\lambda(\xi t)dt + \int_1^\infty t^{1-\lambda+\nu}h_2(t)J_\lambda(\xi t)dt\right\}, \tag{6.8}$$

where λ is given by equation (6.7) and the functions $h_1(t)$, $h_2(t)$ are given by the equations

$$h_1(t) = 2^{-\omega}\,t^{\omega}\, I_{\frac{1}{2}\mu,\ \lambda-\mu}\,F(t), \quad h_2(t) = K_{\frac{1}{2}\mu+\frac{1}{2}\omega,\ \nu-\lambda}G(t). \tag{6.9}$$

We can evaluate the I-integral by means of equation (3.1) provided that

$$\lambda > \mu > -1 \tag{6.10a}$$

and the K-integral by means of equation (3.2) provided that

$$\mu + \omega > -1, \qquad \nu > \lambda\,. \tag{6.10b}$$

When these conditions are satisfied we have the expressions

$$h_1(t) = \frac{2^{1-\omega}t^{\nu+2\omega}}{\Gamma(\lambda-\mu)}\int_0^t (t^2 - \tau^2)^{\lambda-\mu-1}\,\tau^{\mu+1}\,F(\tau)d\tau\,, \tag{6.11}$$

$$h_2(t) = \frac{2t^{\mu+\omega}}{\Gamma(\nu-\lambda)}\int_t^\infty (\tau^2 - t^2)^{\nu-\lambda-1}\,\tau^{1-\nu}G(\tau)d\tau\,,$$

for the component parts of the function h(t).

When the conditions (6.10) or (6.11) are not satisfied the equations (6.6) still furnish a solution of the dual integral equations but the I- and K-fractional integrals have to be interpreted in the manner outlined in § 3.1. We shall not consider the procedure since it will be illustrated below when we discuss Titchmarsh's solution.

6.2 Titchmarsh's solution.

Titchmarsh's solution [cf. Titchmarsh, (1937), p. 334] is essentially the solution (6.6) in the case when $\mu = \nu$, $\beta = 0$ so that the pair of dual integral equations under consideration is

$$\int_0^\infty \xi^{-2\alpha}\,\varphi(\xi)J_\nu\,(\xi x)d\xi = F(x), \qquad x \in I_1, \tag{6.13}$$

$$\int_0^\infty \xi^{-2\beta}\,\varphi(\xi)J_\nu\,(\xi x)d\xi = G(x), \qquad x \in I_2. \tag{6.14}$$

If we put $\mu = \nu$, $\beta = 0$, $\lambda = \nu-\alpha$ into equations (6.6) we find that the solution of these equations is

$$\varphi(\xi) = 2^{-\alpha}\,\xi^{1+\alpha}\left\{\int_0^1 t^{1+\alpha}\,h_1(t)J_{\nu-\alpha}(\xi t)dt + \int_1^\infty t^{1+\alpha}\,h_2(t)J_{\nu-\alpha}(\xi t)dt\right. . \tag{6.15}$$

where $h_1(t)$ and $h_2(t)$ are given by the equations

$$h_1(t) = 2^{2\alpha}\,t^{-2\alpha}\,I_{\frac{1}{2}\nu\,,-\alpha}F(t), \quad h_2(t) = K_{\frac{1}{2}\nu\,-\alpha,\,\alpha}G(t). \tag{6.16}$$

In the computation of $h_1(t)$ and $h_2(t)$ two cases must be distinguished according as α is positive or negative.

<u>Case (i)</u>: $\underline{\alpha} < 0$: We suppose that <u>n</u> is the smallest positive integer for which $n \geqslant -\alpha$. If $\alpha < 0$, then by equations (3.1) and (6.16) we have

$$h_1(t) = \frac{2^{1+2\alpha}t^{-\nu}}{\Gamma(-\alpha)}\int_0^t (t^2-\tau^2)^{-\alpha-1}\,\tau^{\nu+1}\,F(\tau)d\tau\,, \quad (\alpha < 0,\ \nu > -1)$$

and by equations (3.25) and (6.16)

$$h_2(t) = \frac{2(-1)^n}{\Gamma(\alpha+n)}\,t^{\nu-2\alpha-1}\mathcal{D}_t^n\,t\int_t^\infty (\tau^2-t^2)^{\alpha+n-1}\,\tau^{1-\nu}\,G(\tau)d\tau\,.$$

Substituting these expressions into equation (6.15) we obtain the solution

$$\varphi(\xi) = \frac{2^{1+\alpha}\,\xi^{1+\alpha}}{\Gamma(-\alpha)}\int_0^1 t^{1-\nu+\alpha}\,J_{\nu-\alpha}(\xi t)dt\int_0^t (t^2-\tau^2)^{-\alpha-1}\,\tau^{\nu+1}\,F(\tau)d\tau$$

$$+\ (-1)^n\,\frac{2^{1-\alpha}\,\xi^{1+\alpha}}{\Gamma(\alpha+n)}\int_0^1 t^{\nu-\alpha}\,J_{\nu-\alpha}(\xi t)dt\,\mathcal{D}_t^n\,t\int_t^\infty (\tau^2-t^2)^{\alpha+n-1}\,\tau^{1-\nu}G(\tau)d\tau \tag{6.17}$$

valid for $-n < \alpha < 0$, $\nu > -1$.

In the special case in which $G(\tau) = 0$ we obtain the expression

$$\varphi(\xi) = \frac{(2\,\xi)^{1+\alpha}}{\Gamma(-\alpha)}\int_0^1 t^{1-\alpha}\,J_{\nu-\alpha}(\xi t)dt\int_0^1 (1-\zeta^2)^{-\alpha-1}\,\zeta^{\nu+1}\,G(t\zeta)d\zeta, \tag{6.18}$$

valid for $\alpha < 0$; this is Titchmarsh's solution.

In the special case in which $F(\tau) = 0$ and $-1 < \alpha < 0$ we find, on putting $F(\tau) = 0$, $n = 1$ in equation (6.17) that

$$\varphi(\xi) = -\frac{2^{-\alpha}\,\xi^{1+\alpha}}{\Gamma(1+\alpha)}\int_0^1 t^{\nu-\alpha}\,J_{\nu-\alpha}(\xi t)dt\,.\,\frac{d}{dt}\int_t^{\infty}(\tau^2-t^2)^{\alpha}\,\tau^{1-\nu}G(\tau)d\tau\,. \tag{6.19}$$

This is the form of solution obtained by Noble (1958), Williams (1962) and Lowengrub and Sneddon (1962).

<u>Case (ii)</u>: $\alpha > 0$. If $n > \alpha > 0$ the, by equations (3.21) and (6.16) we have

$$h_1(t) = \frac{2^{1+2\alpha}\,t^{-\nu-1}}{\Gamma(n-\alpha)}\mathcal{D}_t^n\,t\int_0^t (t^2-\tau^2)^{n-\alpha-1}\,\tau^{\nu+1}\,F(\tau)d\tau$$

Also using equations (3.2) and (6.16) we have

$$h_2(t) = \frac{2t^{\nu-2\alpha}}{\Gamma(\alpha)}\int_t^{\infty}(\tau^2-t^2)^{\alpha-1}\,\tau^{-\nu+1}\,G(\tau)d\tau\,.$$

Substituting these expressions into equation (6.15) we obtain the solution

$$\varphi(\xi) = \frac{2^{1+\alpha}\,\xi^{1+\alpha}}{\Gamma(n-\alpha)}\int_0^1 t^{-\nu+\alpha}\,J_{\nu-\alpha}(\xi t)dt\,\mathcal{D}_t^n\,t\int_0^t (t^2-\tau^2)^{n-\alpha-1}\,\tau^{\nu+1}F(\tau)d\tau$$

$$+\frac{2^{1-\alpha}\,\xi^{1+\alpha}}{\Gamma(\alpha)}\int_1^{\infty}t^{1+\nu-\alpha}\,J_{\nu-\alpha}(\xi t)dt\int_t^{\infty}(\tau^2-t^2)^{\alpha-1}\,\tau^{-\nu+1}\,G(\tau)d\tau \tag{6.19}$$

The special case $0 < \alpha < 1$ is of interest. Putting $n = 1$ in equation (6.19) we obtain the solution

$$\varphi(\xi) = \frac{2^{\alpha}\,\xi^{1+\alpha}}{\Gamma(1-\alpha)}\int_0^1 t^{-\nu+\alpha}\,J_{\nu-\alpha}(\xi t)dt\,\frac{d}{dt}\,t^{2-2\alpha+\nu}\int_0^1 (1-\zeta^2)^{-\alpha}\,\zeta^{\nu+1}F(\zeta t)d\zeta$$

$$+\frac{2^{1-\alpha}\,\xi^{1+\alpha}}{\Gamma(1-\alpha)}\int_1^{\infty}t^{1+\alpha}\,J_{\nu-\alpha}(\xi t)dt\int_1^{\infty}(\zeta^2-1)^{\alpha-1}\,\zeta^{-\nu+1}\,G(\zeta t)d\zeta \tag{6.20}$$

Since

$$\frac{d}{dt}\left[t^{-\nu+\alpha}\,J_{\nu-\alpha}(\xi t)\right] = -\xi t^{-\nu+\alpha}\,J_{\nu-\alpha+1}(\xi t)$$

we find on integrating by parts in the first term that if $2+\nu-2\alpha > 0$

$$\varphi(\xi) = \frac{2^{\alpha}\xi^{2\alpha}}{\Gamma(1-\alpha)}\left\{\xi^{1-\alpha} J_{\nu-\alpha}(\xi)\int_0^1 (1-\zeta^2)^{-\alpha}\zeta^{\nu+1}F(\zeta)d\zeta\right.$$

$$\left. + \int_0^1 (1-\zeta^2)^{-\alpha}\ \zeta^{\nu+1}\,d\zeta\int_0^1 (\xi t)^{2-\alpha} J_{\nu-\alpha+1}(\xi t)F(\zeta t)dt\right\}$$

$$+ \frac{2^{1-\alpha}\xi^{1+\alpha}}{\Gamma(\alpha)}\int_1^{\infty} t^{1+\alpha} J_{\nu-\alpha}(\xi t)dt\int_1^{\infty}(\zeta^2-1)^{\alpha-1}\zeta^{-\nu+1}G(\zeta t)d\zeta \tag{6.21}$$

If we put $G(t) = 0$ in this equation we obtain Busbridge's solution [Busbridge (1938)]. On the other hand if we put $F(t) = 0$ in equation (6.19) we get the solution derived previously by Noble (1955), Williams (1962) and Lowengrub and Sneddon (1962).

6.3 Noble's Solution.

Noble (1958) gave a solution of the equations (6.4) with $\mu = \nu$, namely of the equations

$$\begin{aligned} S_{\frac{1}{2}\nu-\alpha,2\alpha}\,\psi(x) &= f(x), \qquad x \in I_1 \\ S_{\frac{1}{2}\nu-\beta,2\beta}\,\psi(x) &= g(x), \qquad x \in I_2 \end{aligned} \tag{6.23}$$

which is based virtually on the same computation. Since

$$I_{\frac{1}{2}\nu+\alpha,\beta-\alpha}\, S_{\frac{1}{2}\nu-\alpha,2\alpha} = K_{\frac{1}{2}\nu-\alpha,\alpha-\beta}\, S_{\frac{1}{2}\nu-\beta,2\beta}$$

(both being equal to $S_{\frac{1}{2}\nu-\alpha,\alpha+\beta}$) we can write the equations (6.23) in the form

$$I_{\frac{1}{2}\nu+\alpha,\beta-\alpha} f = K_{\frac{1}{2}\nu-\alpha,\alpha-\beta}\, g$$

or in the form

$$K_{\frac{1}{2}\nu-\alpha,\alpha-\beta}\, g_1 - I_{\frac{1}{2}\nu+\alpha,\beta-\alpha}\, f_2 = \Phi_1 \tag{6.24}$$

where

$$\Phi_1 = I_{\frac{1}{2}\nu+\alpha,\beta-\alpha}\, f_1 - K_{\frac{1}{2}\nu-\alpha,\alpha-\beta}\, g_2. \tag{6.25}$$

Now it will be recalled that f_1 and g_2 are prescribed functions but that f_2 and g_1 are not known.

If we evaluate equation (6.24) on I_1 we obtain the integral equation

$$K_{\frac{1}{2}\nu-\alpha,\alpha-\beta}\, g_1 = \Phi_1, \quad x \in I_1 \tag{6.26}$$

which determines g_1. Hence $g = g_1 + g_2$ is completely determined and may be obtained by means of (4.5) in the form

$$\psi = S_{\frac{1}{2}\nu+\beta,-2\beta}\, g. \tag{6.27}$$

On the other hand, if we evaluate equation (6.24) on I_2 we obtain the integral equation

$$I_{\frac{1}{2}\nu+\alpha,\beta-\alpha}\, f_2 = -\Phi_1, \quad x \in I_2 \tag{6.28}$$

by means of which we may determine f_2. Hence $f = f_1 + f_2$ is completely determined and ψ may be found, from equation (4.5), in the form

$$\psi = S_{\frac{1}{2}\nu+\alpha,-2\alpha}\, f. \tag{6.29}$$

To illustrate the method we shall consider two special cases:-

<u>Case (i)</u>: $0 < \alpha < 1$, $\beta = 0$.

In this case equation (6.26) takes the form

$$K_{\frac{1}{2}\nu-\alpha,\alpha}\, g_1 = \Phi_1, \quad x \in I_1$$

which is equivalent to

$$\int_x^1 (t^2 - x^2)^{\alpha-1}\, t^{1-2\nu}\, g_1(t)\,dt = \tfrac{1}{2}\,\Gamma(\alpha)\, x^{2\alpha-\nu}\, \Phi_1(x) \tag{6.30}$$

$$\begin{aligned}
\Phi_1(x) &= I_{\frac{1}{2}\nu+\alpha,-\alpha}\, f_1(x) - K_{\frac{1}{2}\nu-\alpha,\alpha}\, g_2(x) \\
&= \tfrac{1}{2}\, x^{-\nu-1}\, \frac{d}{dx}\, x^{2+\nu}\, I_{\frac{1}{2}\nu+\alpha,1-\alpha}\, f_1(x) - K_{\frac{1}{2}\nu-\alpha,\alpha}\, g_2(x) \\
&= 2^{2\alpha-1}\, x^{-\nu-1}\, \frac{d}{dx}\, x^{2+\nu}\, I_{\frac{1}{2}\nu+\alpha,1-\alpha}\, x^{-2\alpha}\, F_1(x) - K_{\frac{1}{2}\nu-\alpha,\alpha}\, G_2(x) \\
&= 2^{2\alpha-1}\, x^{-\nu-1}\, \frac{d}{dx}\, x^{-2\alpha+\nu+2}\, I_{\frac{1}{2}\nu,1-\alpha}\, F_1(x) - K_{\frac{1}{2}\nu-\alpha,\alpha}\, G_2(x) \\
&= \frac{2^{2\alpha}\, x^{-\nu-1}}{\Gamma(1-\alpha)}\, \frac{d}{dx}\int_0^x (x^2 - u^2)^{-\alpha}\, u^{\nu-1}\, F_1(u)\,du \\
&\qquad - \frac{2x^{2\alpha}}{\Gamma(\alpha)}\int_1^\infty (u^2 - x^2)^{\alpha-1}\, u^{-\nu+1}\, G_2(u)\,du, \quad 0 < x < 1.
\end{aligned} \tag{6.31}$$

Writing down the solution of equation (6.30) by means of the formula (2.17) we find that if $0 < \alpha < 1$

$$g_1(t) = -\frac{t^{2\nu-1}}{\Gamma(1-\alpha)}\frac{d}{dt}\int_t^1 \frac{\tau^{2\alpha-\nu+1}\Phi_1(\tau)d\tau}{(\tau^2-t^2)^{\alpha}}, \quad 0 < t < 1 \qquad (6.32)$$

From equation (6.27) we then have finally that the solution of the dual integral equations

$$\begin{aligned} \int_0^\infty \xi^{-2\alpha}\varphi(\xi)J_\nu(\xi x)d\xi &= F(x), \quad 0 < x < 1 \\ \int_0^\infty \varphi(\xi)J_\nu(\xi x)d\xi &= G(x), \quad x > 1 \end{aligned} \qquad (6.33)$$

$(0 < \alpha < 1)$ is given by the formula

$$\varphi(\xi) = \xi S_{\frac{1}{2}\nu,0}\, g = \xi\int_0^1 tg_1(t)J_\nu(\xi t)dt + \xi\int_1^\infty tG_2(t)J_\nu(\xi t)dt \qquad (6.34)$$

where $g_1(t)$ is given by equations (6.32) and (6.31).

<u>Case (ii)</u>: $\alpha = 0$, $0 < b < 1$.

In this case equation (6.28) takes the form

$$I_{\frac{1}{2}\nu,\beta}\, f_2(x) = -\Phi_1(x), \quad x \in I_2$$

which is equivalent to

$$\int_1^x (x^2-t^2)^{\beta-1} f_2(t)t^{2\beta+1}\,dt = -\tfrac{1}{2}\Gamma(\beta)x^{2\beta+\nu}\Phi_1(x), \quad x > 1 \qquad (6.35)$$

with

$$\begin{aligned} \Phi_1(x) &= I_{\frac{1}{2}\nu,\beta}f_1 - K_{\frac{1}{2}\nu,-\beta}\, g_2 \\ &= I_{\frac{1}{2}\nu,\beta}\, F_1 - 2^{2\beta}x^{-2\beta}K_{\frac{1}{2}\nu+\beta,-\beta}\, G, \qquad x > 1 \\ &= I_{\frac{1}{2}\nu,\beta}\, f_1 - 2^{2\beta-1}x^{\nu-1}\frac{d}{dx}\{x^{2-2\beta-\nu}K_{\frac{1}{2}\nu+\beta-1,1-\beta}\, G(x)\} \\ &= \frac{2x^{-2\beta-\nu}}{(\beta)}\int_0^1 (x^2-u^2)^{\beta-1}u^{\nu+1}F(u)du \\ &\quad - \frac{2^{2\beta}x^{\nu-1}}{\Gamma(1-\beta)}\frac{d}{dx}\int_x^\infty (u^2-x^2)^{-\beta}u^{1-\nu}G(u)du, \quad (x > 1) \end{aligned} \qquad (6.36)$$

The solution of the integral equation (6.35) can be derived by means of equation (2.16) in the form

$$f_2(t) = -\frac{t^{-2\beta-1}}{\Gamma(1-\beta)} \frac{d}{dt} \int_1^t \frac{\tau^{2\beta+\ \ +1}\, \Phi_1(\tau)}{(t^2-\tau^2)^\beta}\, d\tau . \tag{6.37}$$

From equation (6.29) we have finally that the solution of the dual integral equations

$$\begin{aligned} &\int_0^\infty \varphi(\xi) J_\nu(x\xi)\, d\xi = F(x), \quad 0 < x < 1 \\ &\int_0^\infty \xi^{-2\beta} \varphi(\xi) J\ (x\xi)\, d\xi = G(x), \quad x > 1 \end{aligned} \tag{6.38}$$

$(0 < \beta < 1)$ is given by the formula

$$\varphi(\xi) = \xi \int_0^1 tF(t) J_\nu(\xi t)\, dt + \xi \int_1^\infty t f_2(t) J_\nu(\xi t)\, dt \tag{6.39}$$

where $f_2(t)$ is given by equations (6.37) and (6.36).

6.4 Gordon-Copson Solution.

The solution of Gordon (1954) or Copson (1961) is obtained if we regard

$$\psi = S_{\frac{1}{2}\nu+\beta,-\alpha-\beta}\, h \tag{6.40}$$

as a trial solution of the pair of equations (6.23) with unknown h. Substitution of the form (6.40) into the first pair of equations (6.23) yields the result

$$f = S_{\frac{1}{2}\nu-\alpha,2\alpha}\, S_{\frac{1}{2}\nu+\beta,-\alpha-\beta}\, h = I_{\frac{1}{2}\nu+\beta,\alpha-\beta}\, h. \tag{6.41}$$

by (4.8). This is a functional equation for h from which h_1 may be found. Similarly we can derive the equation

$$g = S_{\frac{1}{2}\nu-\beta,2\beta}\, S_{\frac{1}{2}\nu+\beta,-\alpha-\beta}\, h = K_{\frac{1}{2}\nu-\beta,\beta-\alpha}\, h \tag{6.42}$$

for the determination of h_2.

Case (i): $\alpha > \beta$.

In this case h_1 is determined by the integral equation

$$\int_0^x (x^2-u^2)^{\alpha-\beta-1} u^{\nu+2\beta+1} h_1(u)\, du = \tfrac{1}{2}\Gamma(\alpha-\beta) x^{2\alpha+\nu} f(x), \quad x \in I_1 \tag{6.43}$$

while $h_2(x)$ is given (as a result of (3.16) in the form

$$h_2(x) = K_{\frac{1}{2}\nu-\alpha,\alpha-\beta}\, g(x) \quad x \in I_2 \tag{6.45}$$

Taking equations (6.3) into account we see that

$$h_2(x) = 2^{2\beta}\, x^{-2\beta}\, K_{\frac{1}{2}\nu-\alpha+\beta,\alpha-\beta}\, G(x), \quad x \in I_2 \tag{6.46}$$

and that equation (2.16) implies that, if $0 < \alpha-\beta < 1$,

$$h_1(x) = \frac{2^{2\beta}\, x^{-\nu-2\beta-1}}{\Gamma(1-\alpha+\beta)} \frac{d}{dx} \int_0^x \frac{u^{\nu+1}\, F(u)du}{(x^2-u^2)^{\alpha-\beta}}, \quad x \in I_1 \tag{6.47}$$

Writing equation (6.46) in integral form by means of (3.2) we find that

$$h_2(x) = \frac{2^{2\beta+1}\, x^{\nu-2\alpha}}{\Gamma(\alpha-\beta)} \int_x^\infty (u^2-x^2)^{\alpha-\beta-1}\, u^{-\nu+1}\, G(u)du, \quad x \in I_2. \tag{6.48}$$

It follows from equation (6.40) that if $0 < \alpha-\beta < 1$ the solution of the pair of equations (6.13) and (6.14) may be written in the form

$$\varphi(\xi) = 2^{-\alpha-\beta}\, \xi^{\alpha+\beta} \left\{ \int_0^1 x^{1+\alpha+\beta}\, h_1(x) J_\nu(\xi x)dx + \int_1^\infty x^{1+\alpha+\beta}\, h_2(x) J_\nu(\xi x)dx \right\} \tag{6.49}$$

where $h_1(x)$ and $h_2(x)$ are given by equations (6.47) and (6.48).

Case (ii): $\alpha < \beta$.

In this case it follows as a simple application of (3.15) that

$$h_1 = I_{\frac{1}{2}\nu+\alpha,\beta-\alpha}\, f(x), \qquad x \in I_1 \tag{6.50}$$

while $h_2(x)$ is determined by the integral equation

$$\int_x^\infty (u^2-x^2)^{\beta-\alpha-1}\, u^{1+2\alpha-\nu}\, h_2(u)du = \tfrac{1}{2}\Gamma(\beta-\alpha)x^{\beta-\nu}\, g(x), \quad x \in I_2 \tag{6.51}$$

It follows from equation (3.1) that

$$h_1(t) = \frac{2t^{-\nu}}{\Gamma(\beta-\alpha)} \int_0^t (t^2-u^2)^{\beta-\alpha-1}\, u^{\nu+2\alpha+1}\, f(u)du$$

$$= \frac{2^{1+2\alpha}\, t^{-\nu}}{\Gamma(\beta-\alpha)} \int_0^t (t^2-u^2)^{\beta-\alpha-1}\, u^{\nu+1}\, F(u)du, \quad (0 < t < 1), \tag{6.52}$$

and from equation (2.17) that if $0 < \beta-\alpha < 1$,

$$h_2(t) = -\frac{2^{2\beta}\, t^{-2\alpha+\nu-1}}{\Gamma(1+\alpha-\beta)} \frac{d}{dt} \int_t^\infty \frac{u^{1-\beta-\nu}\, G(u)du}{(u^2-t^2)^{\beta-\alpha}}, \quad (t > 1). \tag{6.53}$$

The solution in the case $0 < \beta-\alpha < 1$ is now given by equation (6.49) with $h_1(t)$ and $h_2(t)$ defined by equations (6.52) and (6.53).

7. Dual integral equations occurring in diffraction theory.

We shall now consider two special types of dual integral equations occurring in diffraction theory.

7.1 Peters' solution.

Peters (1961) has considered the pair of dual integral equations

$$\int_k^\infty \xi^{-\mu-\nu} \varphi(\xi) J_\mu(\xi x) dx = f(x), \quad 0 \le x < 1 \tag{7.1}$$

$$\int_0^\infty (\xi^2 - k^2)^\alpha \varphi(\xi) J_\nu(\xi x) dx = g(x), \quad x > 1, \tag{7.2}$$

which may be written in the symbolical form

$$\mathcal{H}_\mu \{\xi^{-1-\mu-\nu} \varphi(\xi) H(\xi - k); x\} = f(x), \quad 0 \le x < 1, \tag{7.3}$$

$$\mathcal{H}_\nu \{\xi^{-1} (\xi^2 - k^2)^\alpha \varphi(\xi); x\} = g(x), \quad x > 1. \tag{7.4}$$

If we operate on both sides of equation (7.3) with $P^k_{\mu,-\alpha-\mu-1}$ and make use of equations (5.3), (5.4) we find that equation (7.3) is equivalent to

$$\mathcal{H}_{-\alpha} \{\tau^\alpha (\tau^2 + k^2)^{-\frac{1}{2}\nu-\frac{1}{2}} \varphi[\sqrt{(\tau^2+k^2)}]; x\} = h_1(x), \quad 0 \le x < 1, \tag{7.5}$$

where

$$h_1(x) = k^{-\alpha} x^\alpha P^k_{\mu,-\alpha-\mu-1} f(x), \qquad 0 \le x < 1$$

Writing down the form of the operator $P^k_{\mu,-\alpha-\mu-1}$ as given by equation (5.1) we find that

$$h_1(x) = k^{-\alpha} x^\alpha \int_0^x t^{\mu+1} (x^2-t^2)^{-\frac{1}{2}\alpha-\frac{1}{2}\mu-\frac{1}{2}} I_{-\frac{1}{2}\alpha-\frac{1}{2}\mu-\frac{1}{2}}\{k\sqrt{(x^2-t^2)}\} f(t)dt. \tag{7.6}$$

Similarly if we operate on both sides of equation (7.4) with $Q^{k}_{\nu,-\alpha+\nu-1}$ and make use of equations (5.5) and (5.6) we obtain the equation

$$\mathfrak{H}_{-\alpha}\{\tau^{\alpha}(\tau^2+k^2)^{-\frac{1}{2}\nu-\frac{1}{2}}\phi[\sqrt{(\tau^2+k^2)}];x\} = h_2(x), \quad x > 1 \tag{7.7}$$

where

$$h_2(x) = k^{\alpha-\nu+1}x^{-\alpha}Q^{k}_{\nu,-\alpha+\nu-1}g(x), \qquad x > 1$$

$$= k^{\alpha-\nu+1}x^{-\alpha}\int_x^\infty t^{1-\nu}J_{-\alpha+\nu-1}\{k\sqrt{(t^2-x^2)}\}(t^2-x^2)^{-\frac{1}{2}\alpha+\frac{1}{2}\nu-1}g(t)dt \tag{7.8}$$

From equations (7.5) and (7.6) we see that

$$\mathfrak{H}_{-\alpha}\{\tau^{\alpha}(\tau^2+k^2)^{-\frac{1}{2}\nu-\frac{1}{2}}\phi[\sqrt{(\tau^2+k^2)}];x\} = h(x) \tag{7.9}$$

where $h(x) = h_1(x)$, $0 \le x < 1$ and $h(x) = h_2(x)$, $x > 1$. Inverting equation (7.9) by means of the Hankel inversion theorem we have that

$$\tau^{\alpha}(\tau^2+k^2)^{-\frac{1}{2}\nu-\frac{1}{2}}\phi[\sqrt{(\tau^2+k^2)}]$$

$$= \int_0^\infty xh(x)J_{-\alpha}(x\tau)dx$$

$$= \int_0^1 xh_1(x)J_{-\alpha}(x\tau)dx + \int_1^\infty xh_2(x)J_{-\alpha}(x\tau)dx$$

Making the sbustitution $\tau = \sqrt{(\xi^2 - k^2)}$ we finally obtain the solution

$$\phi(\xi) = \xi^{\nu+1}(\xi^2-k^2)^{\frac{1}{2}\alpha}\Big\{\int_0^1 xh_1(x)J_{-\alpha}\{x\sqrt{(\xi^2-k^2)}\}dx$$

$$+ \int_1^\infty xh_2(x)J_{-\alpha}\{x\sqrt{(\xi^2-k^2)}\}dx \tag{7.10}$$

where $h_1(x)$ and $h_2(x)$ are given respectively by equations (7.6) and (7.8). The integrals defining $h_1(x)$ and $h_2(x)$ will exist only if

$$\nu > -\alpha > \mu \tag{7.11}$$

7.2 Ahiezer's solution

Ahiezer (1954) has considered the pair of dual integral equations

$$\int_k^\infty \varphi(\xi) J_o(x\xi) d\xi = f(x), \qquad 0 < x < 1 \tag{7.12}$$

$$\int_0^\infty (\xi^2 - k^2)^\alpha \varphi(\xi) J_o(x\xi) d\xi = 0, \; x > 1, \tag{7.13}$$

with $-1 < \alpha < 1$. These equations can be written in the symbolic forms

$$\mathcal{H}_o \{\xi^{-1} \varphi(\xi) H(\xi - k); x\} = f(x), \qquad 0 < x < 1$$

$$\mathcal{H}_o \{\xi^{-1} (\xi^2 - k^2)^\alpha \varphi(\xi); x\} = 0, \qquad x > 1.$$

If we act on both sides of the first of these equations with the operator $P^k_{o,-\alpha}$ and on both sides of the second one with the operator $Q^k_{o,-1-\alpha}$ and make use of equations (5.3) and (5.5) we find that these equations are equivalent to the relations

$$\mathcal{H}_{1-\alpha} \{\tau^{-1+\alpha} (\tau^2 + k^2)^{-\frac{1}{2}} \varphi[\sqrt{(\tau^2 + k^2)}]; x\} = k^\alpha x^{-1+\alpha} P^k_{o,-\alpha} f(x),$$

$$0 < x < 1, \tag{7.14}$$

$$\mathcal{H}_{-\alpha} \{\tau^\alpha (\tau^2 + k^2)^{-\frac{1}{2}} \varphi[\sqrt{(\tau^2 + k^2)}]; x\} = 0, \qquad x > 1.$$

respectively.

Now, since we have the relation

$$\frac{d}{dx} x^{1-\alpha} J_{1-\alpha}(x\tau) = x^{1-\alpha} \tau J_{-\alpha}(x\tau)$$

we see that the second of these equations is equivalent to the equation

$$\frac{d}{dx} x^{1-\alpha} \mathcal{H}_{1-\alpha} \{\tau^{-1+\alpha} (\tau^2 + k^2)^{-\frac{1}{2}} \varphi[\sqrt{(\tau^2 + k^2)}]; x\} = 0$$

which may be integrated to give

$$\mathcal{H}_{1-\alpha} \{\tau^{-1+\alpha} (\tau^2 + k^2)^{-\frac{1}{2}} \varphi[\sqrt{(\tau^2 + k^2)}]; x\} = cx^{\alpha-1}, \quad x > 1, \tag{7.15}$$

where <u>c</u> is an arbitrary constant. Applying the Hankel inversion theorem to equations (7.14) and (7.15) we obtain the solution

$$\tau^{-1+\alpha} (\tau^2 + k^2)^{-\frac{1}{2}} \varphi(\sqrt{\tau^2 + k^2}) = c \int_1^\infty x^\alpha J_{1-\alpha}(x\tau) dx$$

$$+ \int_0^1 x J_o(\tau x) P^k_{o,-\alpha} f(x) dx. \tag{7.16}$$

Now

$$\int_1^x x^\alpha J_{1-\alpha}(x\tau)d\tau = \frac{1}{\alpha}[x^\alpha J_{-\alpha}(\tau x) - J_{-\alpha}(\tau)].$$

If $-1 < \alpha < \frac{1}{2}$,

$$\lim_{x\to\infty} x^\alpha J_{-\alpha}(x) = 0$$

and we obtain the solution

$$\tau^{-1+\alpha}(\tau^2+k^2)^{-\frac{1}{2}}\varphi(\sqrt{\tau^2+k^2}) = \int_0^1 xJ_0(\tau x)P^k_{0,-\alpha}f(x) - \frac{c}{\alpha}J_{-\alpha}(\tau).$$

Writing out the operator $P^k_{0,-\alpha}$ we obtain the solution

$$\tau^{-1+\alpha}(\tau^2+k^2)^{-\frac{1}{2}}\varphi(\sqrt{\tau^2+k^2}) = \int_0^1 xJ_0(\tau x)dx\int_0^x t^{\eta+1}I_\alpha\{k\sqrt{(x^2-t^2)}\}(x^2-t^2)^{\frac{1}{2}\alpha}f(t)dt - \frac{c}{\alpha}J_{-\alpha}(\tau)$$

(7.17)

If $\frac{1}{2} < \alpha < 1$, then for φ to be finite we must have $c = 0$. This condition is usually equivalent to a physical condition which is easily recognizable.

8. Dual Integral Equations with General Weight Function.

The technique developed in § 6 enables us to discuss the slightly more general dual integral equations

$$\int_0^\infty \xi^{-2\alpha}[1+k(\xi)]\varphi(\xi)J_\nu(\xi x)d\xi = F(x), \quad x \in I_1 \tag{8.1}$$

$$\int_0^\infty \xi^{-2\beta}\varphi(\xi)J_\nu(\xi x)d\xi = G(x), \quad x \in I_2 \tag{8.2}$$

which with the substitutions (6.3) may be written in the form

$$S_{\frac{1}{2}\nu-\alpha,2\alpha}(1+k)\psi = f, \quad S_{\frac{1}{2}\nu-\beta,2\beta}\psi = g, \tag{8.3}$$

where f_1, g_2 and k are known. The solution we give here is essentially that due to Cooke (1956). We again use

$$\psi = S_{\frac{1}{2}\nu+\beta,-\alpha-\beta}h \tag{8.4}$$

with unknown h as a trial solution. Substituting from (8.4) into (8.3) we find that

$$S_{\frac{1}{2}\nu-\alpha,2\alpha}S_{\frac{1}{2}\nu+\beta,-\alpha-\beta}h + S_{\frac{1}{2}\nu-\alpha,2\alpha}kS_{\frac{1}{2}\nu+\beta,-\alpha-\beta}h = f, \tag{8.5}$$

$$S_{\frac{1}{2}\,\nu-\beta,2\beta}\,S_{\frac{1}{2}\,\nu+\beta,-\alpha-\beta}h = g. \tag{8.6}$$

Using the relations (4.9) and (3.16) we obtain the equation

$$h_2 = K_{\frac{1}{2}\,\nu-\alpha,\alpha-\beta}g_2, \quad \text{on } I_2, \tag{8.7}$$

determining h on I_2. Evaluating equation (8.5) on I_1 we obtain the integral equation

$$I_{\frac{1}{2}\,\nu+\beta,\alpha-\beta}h_1 + S_{\frac{1}{2}\,\nu-\alpha,2\alpha}k\,S_{\frac{1}{2}\,\nu+\beta,-\alpha-\beta}h_1$$

$$= f - S_{\frac{1}{2}\,\nu-\alpha,2\alpha}k\,S_{\frac{1}{2}\,\nu+\beta,-\alpha-\beta}h_2, \quad \text{on } I_1$$

Using equations (3.15) and (4.6) we may rewrite this equation in the form

$$h_1 + S_{\frac{1}{2}\,\nu-\alpha,\alpha+\beta}k\,S_{\frac{1}{2}\,\nu+\beta,-\alpha-\beta}h_1 = \Phi_1 \quad \text{on } I_1 \tag{8.8}$$

where

$$\Phi_1 = I_{\frac{1}{2}\,\nu+\alpha,\beta-\alpha}f - S_{\frac{1}{2}\,\nu-\alpha,\alpha+\beta}k\,S_{\frac{1}{2}\,\nu+\beta,-\alpha-\beta}h_2. \tag{8.9}$$

Now

$$S_{\frac{1}{2}\,\nu-\alpha,\alpha+\beta}k\,S_{\frac{1}{2}\,\nu+\beta,-\alpha-\beta}h_1(x)$$

$$= 2^{\alpha+\beta}x^{-\alpha-\beta}\int_0^\infty t^{1-\alpha-\beta}J_{\nu+2\beta-\alpha}(xt)k(t)dt\; 2^{-\alpha-\beta}t^{\alpha+\beta}\int_0^1 u^{1+\alpha+\beta}h_1(u)J_{\nu+\alpha-\beta}(ut)du$$

Interchanging the orders of the integrations we find that this integral can be written in the form

$$\int_0^1 h_1(u)K(x, u)du$$

where

$$K(x, u) = u\left(\frac{u}{x}\right)^{\alpha+\beta}\int_0^\infty tk(t)J_{\nu+\beta-\alpha}(xt)J_{\nu+\alpha-\beta}(ut)dt$$

Equation (8.8) now takes the form

$$h_1(x) + \int_0^1 K(x, u)h_1(u)du = \Phi_1(x), \quad x \in I_1,$$

where $\Phi_1(x)$, defined by equation (8.9), is a known function of $\underline{x}$. The problem is thus reduced to the solution of an integral equation of the second kind of Fredholm's type.

Table I

Relations satisfied by the Erdelyi-Kober Operators and the Modified Operator of Hankel Transforms

$$I_{\eta,\alpha}\, I_{\eta+\alpha,\beta} = I_{\eta,\alpha+\beta} \qquad (3.9)$$

$$K_{\eta,\alpha}\, K_{\eta+\alpha,\beta} = K_{\eta,\alpha+\beta} \qquad (3.11)$$

$$I^{-1}_{\eta,\alpha} = I_{\eta+\alpha,-\alpha} \qquad (3.15)$$

$$K^{-1}_{\eta,\alpha} = K_{\eta+\alpha,-\alpha} \qquad (3.16)$$

$$I_{\eta,-n} f(x) = x^{2n-2\eta-1}\, \mathcal{D}^n_x\, x^{2\eta+1} f(x) \qquad (3.20)$$

$$K_{\eta,-n} f(x) = (-1)^n\, x^{2\eta-1}\, \mathcal{D}^n_x\, x^{2n-2\eta+1} f(x) \qquad (3.24)$$

$$I_{\eta,\alpha} f(x) = x^{-2\alpha-2\eta-1}\, \mathcal{D}^n_x\, x^{2n+2\alpha+2\eta+1}\, I_{\eta,\alpha+n} f(x) \qquad (3.21)$$

$$K_{\eta,\alpha} f(x) = (-1)^n\, x^{2\eta-1}\, \mathcal{D}^n_x\, x^{2n-2\eta+1}\, K_{\eta-n,\alpha+n} f(x) \qquad (3.25)$$

$$S^{-1}_{\eta,\alpha} = S_{\eta+\alpha,-\alpha} \qquad (4.5)$$

$$I_{\eta+\alpha,\beta}\, S_{\eta,\alpha} = S_{\eta,\alpha+\beta} \qquad (4.6)$$

$$K_{\eta,\alpha}\, S_{\eta+\alpha,\beta} = S_{\eta,\alpha+\beta} \qquad (4.7)$$

$$S_{\eta+\alpha,\beta}\, S_{\eta,\alpha} = I_{\eta,\alpha+\beta} \qquad (4.8)$$

$$S_{\eta,\alpha}\, S_{\eta+\alpha,\beta} = K_{\eta,\alpha+\beta} \qquad (4.9)$$

$$S_{\eta+\alpha,\beta}\, I_{\eta,\alpha} = S_{\eta,\alpha+\beta} \qquad (4.10)$$

$$S_{\eta,\alpha}\, K_{\eta+\alpha,\beta} = S_{\eta,\alpha+\beta} \qquad (4.11)$$

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